Inequalities - Part 2

SOURANGSHU GHOSH

In Part 1 of this series of articles, we had considered the arithmetic mean-geometric mean inequality along with some variants and some applications. Now in Part 2, we look at the Cauchy-Schwartz inequality and some applications.

The Cauchy Schwartz Inequality

An important result often used in mathematical Olympiads is the Cauchy-Schwartz inequality. It is especially useful in proving inequalities that have cyclic and symmetric terms. It can also be used to prove many established inequalities. Here is the statement of the result.

Theorem 1 (Cauchy-Schwartz inequality). **If** $a_1, a_2, \ldots, a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$, all numbers being real, the following inequality holds

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2, \tag{1}$$

with equality precisely when the sequences are proportional to each other, i.e.,

$$a_1 = tb_1, \quad a_2 = tb_2, \quad \ldots, \quad a_n = tb_n \tag{2}$$

for some real number t.

Proof. The proof of the theorem is from [1]. Recall the AM-GM inequality for two positive numbers x, y: it states that $x + y \geq 2\sqrt{xy}$, with equality if and only if $x = y$. We apply this as shown below.

Let $A = a_1^2 + a_2^2 + \cdots + a_n^2$, $B = b_1^2 + b_2^2 + \cdots + b_n^2$. For $i = 1, 2, \ldots, n$, we have:

$$\frac{a_i^2}{A} + \frac{b_i^2}{B} \geq \frac{2a_i b_i}{\sqrt{AB}}.$$

Summing the n inequalities, we get:

$$\frac{A}{A} + \frac{B}{B} \geq \frac{2\left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)}{\sqrt{AB}},$$

$$\therefore \quad \sqrt{AB} \geq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

Keywords: Inequality, AM-GM inequality, Cauchy-Schwartz inequality, Holder's inequality.

i.e.,

$$\left(a_1^2+a_2^2+\cdots+a_n^2\right)\cdot\left(b_1^2+b_2^2+\cdots+b_n^2\right)\geq\left(a_1b_1+a_2b_2+\cdots+a_nb_n\right)^2.$$

Equality holds if and only if for each subscript i we have:

$$\frac{a_i^2}{A}=\frac{b_i^2}{B},$$

i.e., $a_i = tb_i$ where $t=\sqrt{A}/\sqrt{B}$. □

Theorem 2 (Alternative form of Cauchy-Schwartz)**.** If $a_1, a_2, \ldots, a_n > 0$ and $b_1, b_2, \ldots, b_n > 0$, all numbers being real, the following inequality holds

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}. \tag{3}$$

Moreover, equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$.

Proof of the alternative form. The proof is from [1]. We proceed inductively, starting with the case $n = 2$. This specific case may be stated as follows: If $a, b > 0$ and $x, y > 0$, all numbers being real, then

$$\frac{a^2}{x}+\frac{b^2}{y}\geq\frac{(a+b)^2}{x+y},$$

with equality if and only if $a/x = b/y$. To prove this, we bring all the terms to one side and multiply through by $xy(x+y)$, which is a positive quantity. We obtain:

$$\begin{aligned}(x+y)\left(a^2y+b^2x\right)-xy(a+b)^2&=a^2y^2-2abxy+b^2x^2\\&=(ay-bx)^2\geq 0,\end{aligned}$$

with equality if and only if $ay - bx = 0$, i.e., $a/x = b/y$.

Now consider the case $n = 3$. The result may be stated as follows: If $a, b, c > 0$ and $x, y, z > 0$, all numbers being real, then

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\geq\frac{(a+b+c)^2}{x+y+z},$$

with equality if and only if $a/x = b/y = c/z$. To prove this, we use the result just proved (for the case $n = 2$). We have:

$$\begin{aligned}\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}&\geq\frac{(a+b)^2}{x+y}+\frac{c^2}{z}\\&\geq\frac{(a+b+c)^2}{x+y+z}\qquad\text{(by applying the result once again).}\end{aligned}$$

The condition for equality should be clear: $a/x = b/y = c/z$.

The extension to the general case is now a matter of detail; we do not give the steps here. □

Some applications of the Cauchy-Schwartz inequality

We now look at some inequalities that make use of the Cauchy-Schwartz inequality.

Example 1: Nesbitt's inequality. For positive real numbers a, b, c, prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}. \tag{4}$$

We had proved this inequality in the first part of this series, but we give a different approach now.

Proof. The solution is from [1]. Adding 1 to each term, the inequality is transformed to:

$$(a+b+c)\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) \geq \frac{9}{2}.$$

Let $x=\sqrt{b+c}$, $y=\sqrt{c+a}$, $z=\sqrt{a+b}$, so that $x^2+y^2+z^2=2(a+b+c)$. Then the inequality may be rewritten as:

$$\left(x^2+y^2+z^2\right)\cdot\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq 9.$$

This now follows immediately from the Cauchy-Schwartz inequality, for we have:

$$\left(x^2+y^2+z^2\right)\cdot\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \geq \left(x\cdot\frac{1}{x}+y\cdot\frac{1}{y}+z\cdot\frac{1}{z}\right)^2=9.$$

Example 2: A problem from IMO 1995. Prove that for any three positive real numbers a, b, c such that the product $abc=1$,

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}. \tag{5}$$

Proof. The solution is from [1]. We use the alternative form of the Cauchy-Schwartz inequality. Let $x=1/a$, $y=1/b$, $z=1/c$. Then by the given condition we obtain $xyz=1$. Note that

$$\begin{aligned}
&\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\\
&=\frac{1}{(1/x^3)(1/y+1/z)}+\frac{1}{(1/y^3)(1/z+1/x)}+\frac{1}{(1/z^3)(1/x+1/y)}\\
&=\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}.
\end{aligned}$$

Now by the Cauchy-Schwartz inequality (in its 'alternative form'),

$$\begin{aligned}
\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y} &\geq \frac{(x+y+z)^2}{2(x+y+z)}=\frac{x+y+z}{2}\\
&\geq 3\cdot\frac{(xyz)^{1/3}}{2}=\frac{3}{2}.
\end{aligned}$$

Example 3. Prove that for all $a, b, c \geq 1$,

$$\sqrt{a^2-1}+\sqrt{b^2-1}+\sqrt{c^2-1} \leq \frac{ab+bc+ca}{2}. \tag{6}$$

The problem was posted on Leo Giugiuc's *Cut-The-Knot* Math Facebook page by Professor Dorin Marghidanu. The proof was given by C. Nanuti, D.Trailescu, D. Sitaru and L. Giugiuc.

Proof. Let $x = \sqrt{a^2-1}$, $y = \sqrt{b^2-1}$, $z = \sqrt{c^2-1}$; then $x, y, z \geq 0$. By the Cauchy-Schwartz inequality,

$$\sqrt{x^2+1}\sqrt{y^2+1} = \sqrt{x^2+1}\sqrt{1+y^2} \geq x+y.$$

So we have, by addition,

$$\sqrt{x^2+1}\sqrt{y^2+1} + \sqrt{y^2+1}\sqrt{z^2+1} + \sqrt{z^2+1}\sqrt{x^2+1} \geq 2(x+y+z).$$

In other words we have:

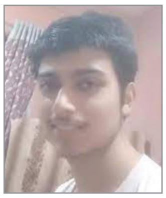
$$ab+bc+ca \geq 2\left(\sqrt{a^2-1}+\sqrt{b^2-1}+\sqrt{c^2-1}\right),$$

which is the required inequality.

Equality holds if and only if $x = y = z = 1$, i.e., if and only if $a = b = c = \sqrt{2}$.

References

1. Riasat, Samin. (2008). Basics of Olympiad Inequalities,
   https://www.researchgate.net/publication/264892067_Basics_of_Olympiad_Inequalities
2. Mildorf, T J, (2005),"Olympiad Inequalities". Preprint, https://artofproblemsolving.com/articles/files/MildorfInequalities.pdf
3. AoPs Online, https://artofproblemsolving.com/wiki/index.php/1964_IMO_Problems/Problem_2
4. AoPs Online, https://artofproblemsolving.com/wiki/index.php/Proofs_of_AM-GM
5. Hardy, G.H.; Littlewood, J.E.; Pólya, G.; "Inequalities"; Cambridge University Press: Cambridge, UK, 1934; ISBN 978-0-521-35880-4.



SOURANGSHU GHOSH is a fourth-year undergraduate student of the Dept of Civil Engg at IIT Kharagpur. He is also pursuing Mathematics as a minor degree. He is interested in Discrete Mathematics and its applications. He has written several articles in mathematics, many in international journals. He has worked on the Reliability of Weld Joints of Nuclear Power Plants in collaboration with IGCAR Kalpakkam, and on the structural reliability of coherent systems, as part of his undergraduate thesis. He enjoys playing the violin in the Indian classical style. He may be contacted at sourangshug123@gmail.com.