Secondary

Student Corner

A Friendly Pair of Triangles

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Terminology. Given a triangle ABC, we say that a triangle PQR is 'inscribed' in ABC if the vertices of PQR lie on the sides of ABC, one vertex per side (e.g., if P lies on side BC, Q lies on side CA, and R lies on side AB). See Figure 1 for an example.

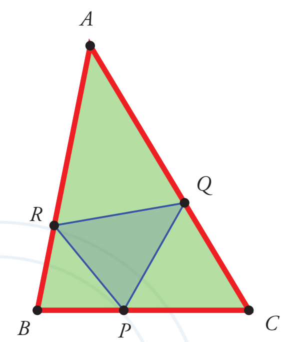


Figure 1.

Two nice inscribed triangles

In this article, which is a continuation of my earlier article [1], I show that for any given triangle ABC, there exist two triangles that are inscribed in it and have the following properties: the two triangles have equal area, and their centroids are equidistant from the centroid of $\triangle ABC$. As the two triangles are related in such a nice way, I call them a "friendly pair" of triangles.

The two triangles are the following.

- Let the incircle of $\triangle ABC$ touch the sides BC, CA, AB at points P, Q, R respectively. Join these three points together to form $\triangle PQR$.
- Let the ex-circle of $\triangle ABC$ opposite vertex A touch side BC at U. Similarly, let the ex-circle of $\triangle ABC$ opposite vertex B touch side CA at V. Finally, let the ex-circle of $\triangle ABC$ opposite vertex C touch side AB at W. Join these three points together to form $\triangle UVW$.

Keywords: Incircle, ex-circle, inscribed triangle, centroid, area formula, position vector

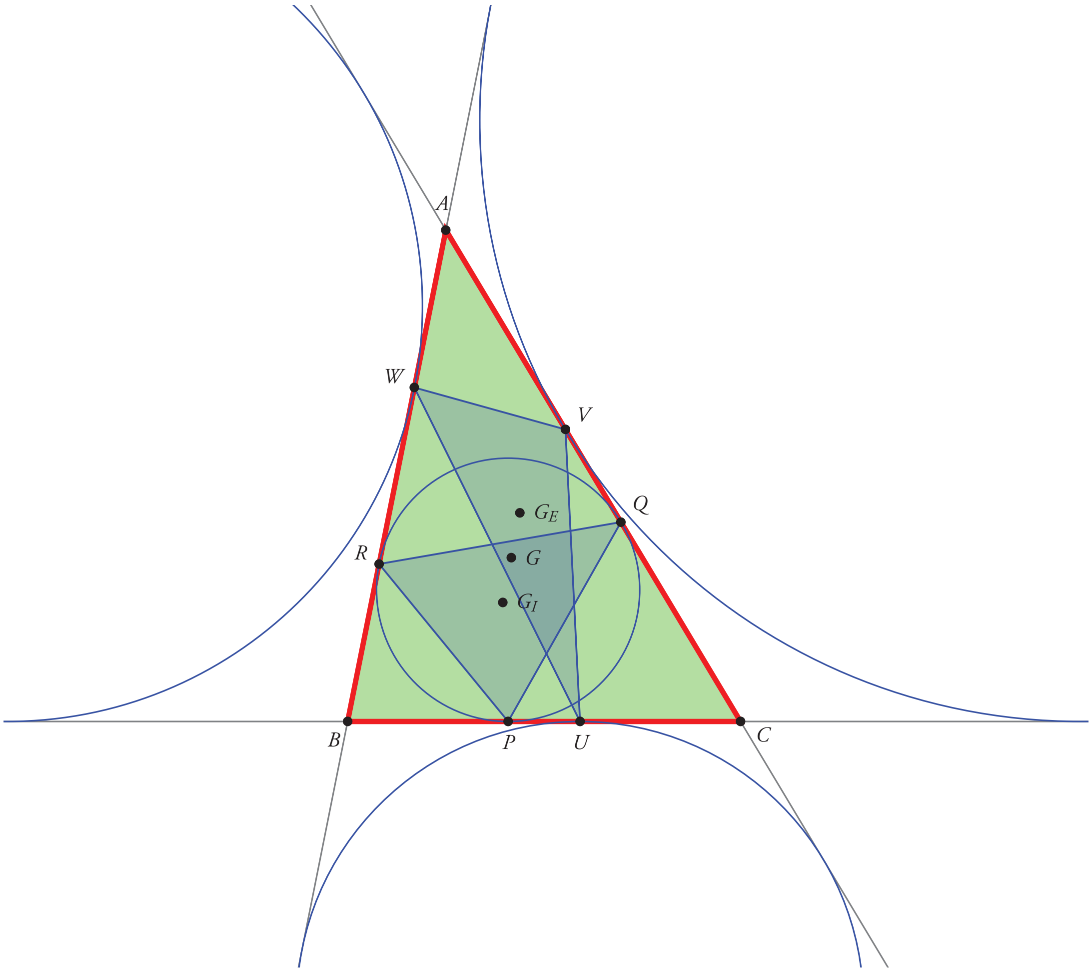


Figure 2.

Then PQR and UVW are a friendly pair of triangles. (See Figure 2.) That is, they have equal area, and their centroids are equidistant from G, the centroid of $\triangle ABC$.

Actually, more can be said. If G_I is the centroid of $\triangle PQR$, and G_E is the centroid of $\triangle UVW$, then G is the midpoint of the segment G_IG_E.

Proofs of the claims

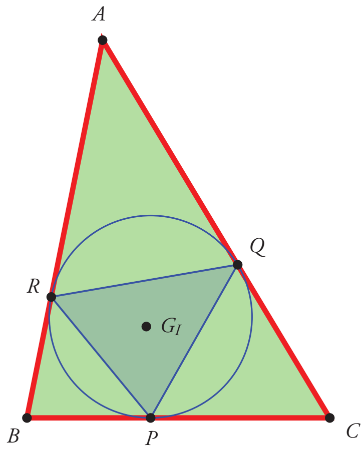
We use standard notation: a, b, c for the sides of the triangle; s for the semi-perimeter; Δ for the area; r for the radius of the incircle; and r_A, r_B, r_C for the radii of the three ex-circles.

Consider first the points where the incircle touches the sides of $\triangle ABC$ (see Figure 3). The lengths of BP, PC, CQ and so on are given at the side.

To see why $AQ = s - a$, denote AQ by x. Then $AR = x$, $BR = c - x$, $BP = c - x$, $CQ = b - x$, $CP = b - x$. Since $BP + CP = a$, we get $(b - x) + (c - x) = a$, so $2x = b + c - a = 2s - 2a$, and $x = s - a$. Similarly for the others.

It follows that:

$$\begin{cases} BP : PC = s - b : s - c, \\ CQ : QA = s - c : s - a, \\ AR : RB = s - a : s - b. \end{cases} \tag{1}$$



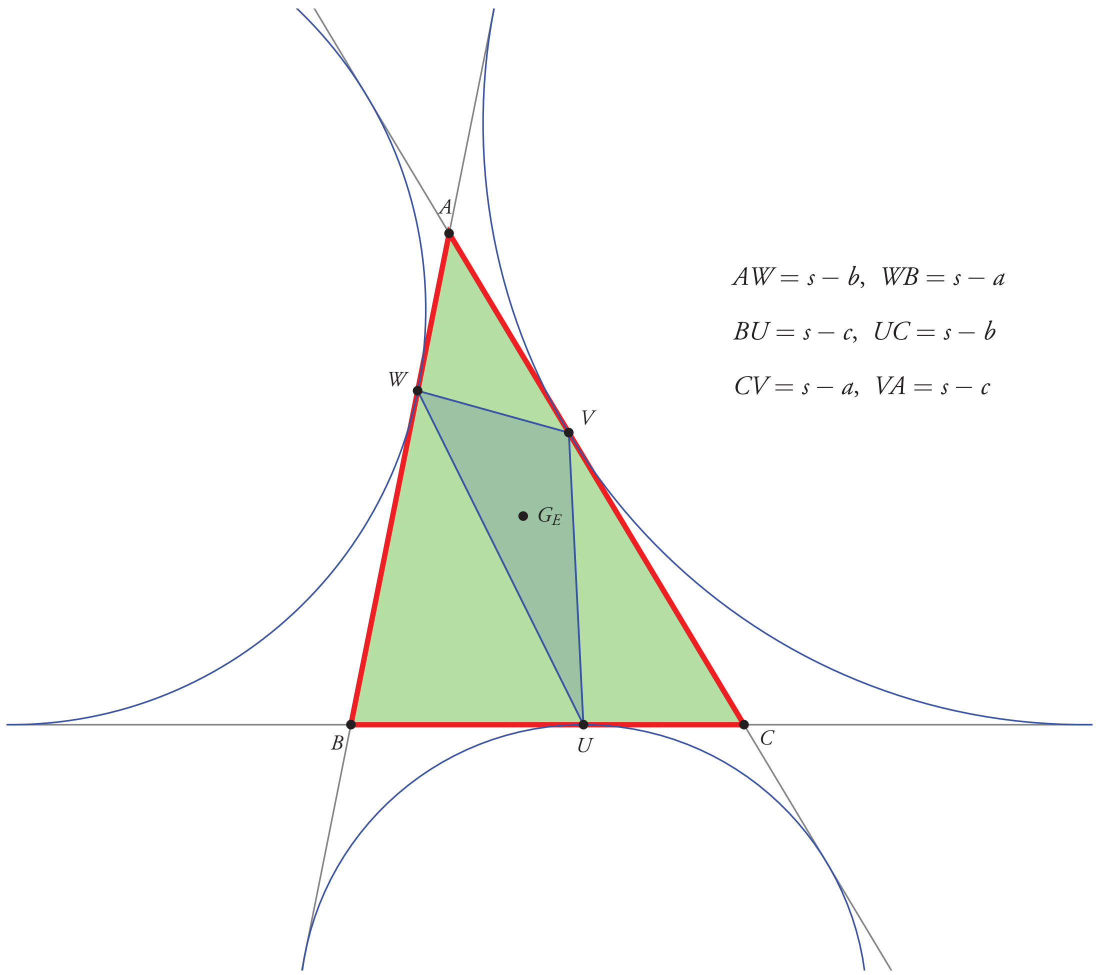
$$AQ = AR = s - a$$

$$BP = BR = s - b$$

$$CP = CQ = s - c$$

Figure 3.

Next, consider the points where the ex-circles touch the sides of $\triangle ABC$ (Figure 4). The lengths of BU, UC, CV and so on are given at the side.



$$AW = s - b, \ \ WB = s - a$$

$$BU = s - c, \ \ UC = s - b$$

$$CV = s - a, \ \ VA = s - c$$

Figure 4.

To see why $BU = s - c$ and $CU = s - b$, note that $AB + BU = AC + CU$, both sides being equal to the length of the tangent from A to the ex-circle opposite vertex A. Hence $c + BU = b + CU$. Also,

$BU + CU = a$. Therefore $c + BU = b + a - BU$, so $2BU = a + b - c = 2s - 2c$, and $BU = s - c$. The other relations follow in the same way.

$$\begin{cases} BU : UC = s - c : s - b, \\ CV : VA = s - a : s - c, \\ AW : WB = s - b : s - a. \end{cases} \tag{2}$$

From (1) and (2) we see that:

- P and U are symmetrically placed on segment BC (i.e., $BP = CU$), so the midpoint D of PU is also the midpoint of BC.
- Q and V are symmetrically placed on segment CA (i.e., $CQ = AV$), so the midpoint E of QV is also the midpoint of CA.
- R and W are symmetrically placed on segment AB (i.e., $AW = BR$), so the midpoint F of RW is also the midpoint of AB.

(Points D, E, F are not marked in the figures.)

The above relations may be expressed using vectors. Denoting the position vector (p.v.) of A by $\mathbf{A}$, the p.v. of B by $\mathbf{B}$, and so on, we have:

$$\begin{cases} \mathbf{P} + \mathbf{U} = \mathbf{B} + \mathbf{C}, \\ \mathbf{Q} + \mathbf{V} = \mathbf{C} + \mathbf{A}, \\ \mathbf{R} + \mathbf{W} = \mathbf{A} + \mathbf{B}. \end{cases} \tag{3}$$

The centroids of $\triangle ABC$, $\triangle PQR$, and $\triangle UVW$ are G, G_I and G_E, respectively. Clearly:

$$\begin{cases} 3\,\mathbf{G} = \mathbf{A} + \mathbf{B} + \mathbf{C}, \\ 3\,\mathbf{G_I} = \mathbf{P} + \mathbf{Q} + \mathbf{R}, \\ 3\,\mathbf{G_E} = \mathbf{U} + \mathbf{V} + \mathbf{W}. \end{cases} \tag{4}$$

It follows from (3) and (4), by addition, that

$$\mathbf{G_I} + \mathbf{G_E} = 2\,\mathbf{G}. \tag{5}$$

Therefore G is the midpoint of the segment connecting G_I and G_E.

Therefore the two centroids are equidistant from G.

Area computations. Now we show that the areas of triangles PQR and UVW are equal. The geometric result we use is the following. Given $\triangle ABC$, let points K and L lie on sides AB and AC respectively (see Figure 5). Then:

$$\frac{\text{Area of } \triangle AKL}{\text{Area of } \triangle ABC} = \frac{AK \cdot AL}{AB \cdot AC}. \tag{6}$$

This is true, because we have:

$$\frac{\text{Area of } \triangle AKL}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}\,AK \cdot AL \cdot \sin A}{\frac{1}{2}\,AB \cdot AC \cdot \sin A} = \frac{AK \cdot AL}{AB \cdot AC}.$$

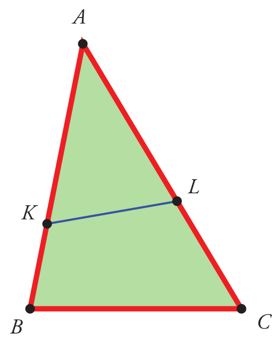


Figure 5.

Applying this result to Figure 3, we get, since the area of $\triangle PQR$ is equal to the area of $\triangle ABC$ minus the area of $\triangle BPR$ minus the area of $\triangle CQP$ minus the area of $\triangle ARQ$:

$$\begin{aligned}\frac{\text{Area of }\triangle PQR}{\text{Area of }\triangle ABC} &= 1 - \frac{(s-b)^2}{ca} - \frac{(s-c)^2}{ab} - \frac{(s-a)^2}{bc} \\ &= 1 - \frac{a(s-a)^2 + b(s-b)^2 + c(s-c)^2}{abc}. \qquad (7)\end{aligned}$$

Similarly applying the result to Figure 4, we get:

$$\begin{aligned}\frac{\text{Area of }\triangle UVW}{\text{Area of }\triangle ABC} &= 1 - \frac{(s-a)(s-c)}{ca} - \frac{(s-a)(s-b)}{ab} - \frac{(s-b)(s-c)}{bc} \\ &= 1 - \frac{(s-a)(s-b)c + (s-b)(s-c)a + (s-c)(s-a)b}{abc}. \qquad (8)\end{aligned}$$

Hence, proving that triangles PQR and UVW have the same area is equivalent to proving that the following two quantities

$$\begin{cases} a(s-a)^2 + b(s-b)^2 + c(s-c)^2, \\ (s-a)(s-b)c + (s-b)(s-c)a + (s-c)(s-a)b, \end{cases} \qquad (9)$$

are identically equal. Since $2s = a + b + c$, this is equivalent to showing that the following two quantities

$$a(b+c-a)^2 + b(c+a-b)^2 + c(a+b-c)^2, \qquad (10)$$

$$(b+c-a)(c+a-b)c + (c+a-b)(a+b-c)a + (a+b-c)(b+c-a)b, \qquad (11)$$

are identically equal. This can be done by directly simplifying both the expressions. But here is another way of proceeding. Look at the coefficients of each kind of term in (10) and (11). In both the expressions, every term has degree 3.

- The coefficient of a^3 is 1 in both (10) and (11). Likewise for b^3 and c^3.
- The coefficient of a^2b is $-2 + 1 = -1$ in both (10) and (11). Likewise for the terms a^2c, b^2a, b^2c, c^2a and c^2b.
- The coefficient of abc is $2 + 2 + 2 = 6$ in both (10) and (11).

As these are the only terms of degree 3 possible, it follows that expressions (10) and (11) are identically equal to each other. Therefore $\triangle PQR$ and $\triangle UVW$ have equal area.

By simplifying these expressions, we can express this area in terms of the elements of the triangle. We use the following identities. (For the proofs, please see the appendix.)

$$\begin{aligned} ab+bc+ca &= r^2+s^2+4Rr, \\ a^2+b^2+c^2 &= 2\left(s^2-r^2-4Rr\right), \\ a^3+b^3+c^3 &= 2s\left(s^2-3r^2-6Rr\right). \end{aligned}$$

Consider the expression $a(s-a)^2+b(s-b)^2+c(s-c)^2$ in (7). Let us simplify it using the above relations. We have, using the short forms $\sum a^3$ for $a^3+b^3+c^3$, $\sum a^2$ for $a^2+b^2+c^2$, and $\sum a$ for $a+b+c$:

$$\begin{aligned} & a(s-a)^2+b(s-b)^2+c(s-c)^2 \\ &= s^2\cdot\sum a-2s\cdot\sum a^2+\sum a^3 \\ &= 2s^3-2s\left(2s^2-2r^2-8Rr\right)+2s\left(s^2-3r^2-6Rr\right) \\ &= 2s\left(s^2-2s^2+2r^2+8Rr+s^2-3r^2-6Rr\right) \\ &= 2s\left(2Rr-r^2\right)=2sr(2R-r), \end{aligned}$$

i.e.,

$$a(s-a)^2+b(s-b)^2+c(s-c)^2=2\Delta(2R-r). \tag{12}$$

The expression $(s-a)(s-b)c+(s-b)(s-c)a+(s-c)(s-a)b$ simplifies to the same quantity, $2\Delta(2R-r)$. Hence:

$$\begin{aligned} \frac{\text{Area of }\triangle PQR}{\text{Area of }\triangle ABC} &= 1-\frac{a(s-a)^2+b(s-b)^2+c(s-c)^2}{abc} \\ &= 1-\frac{2\Delta(2R-r)}{abc}=1-\frac{2\Delta(2R-r)}{4R\Delta} \\ &= 1-\frac{2R-r}{2R}=\frac{r}{2R}. \end{aligned}$$

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Appendix: proofs of some of the background results

Formulas connecting the lengths of the sides of the triangle.

$$ab+bc+ca=r^2+s^2+4Rr, \tag{13}$$

$$a^2+b^2+c^2=2\left(s^2-r^2-4Rr\right), \tag{14}$$

$$a^3+b^3+c^3=2s\left(s^2-3r^2-6Rr\right). \tag{15}$$

Proofs. We take the following relations as well-known:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \Delta = rs, \qquad \Delta = \frac{abc}{4R}.$$

We start with the relation $rs = \sqrt{s(s-a)(s-b)(s-c)}$. Squaring both sides and dividing by s, we get:

$$\begin{aligned} r^2 s &= (s-a)(s-b)(s-c) \\ &= s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc \\ &= s^3 - 2s \cdot s^2 + (ab+bc+ca)s - 4R \cdot rs, \\ \therefore\ ab+bc+ca &= r^2 + s^2 + 4Rr. \end{aligned}$$

Next,

$$\begin{aligned} a^2+b^2+c^2 &= (a+b+c)^2 - 2(ab+bc+ca) \\ &= 4s^2 - 2\left(r^2+s^2+4Rr\right) \\ &= 2\left(s^2 - r^2 - 4Rr\right). \end{aligned}$$

To find an expression for $a^3+b^3+c^3$, we consider the cubic polynomial whose roots are a, b, c. Let the polynomial be

$$x^3 + ux^2 + vx + w. \tag{16}$$

Then we must have the following equalities for the coefficients u, v, w:

$$\begin{aligned} u &= -(a+b+c) = -2s, \\ v &= ab+bc+ca = r^2+s^2+4Rr, \\ w &= -abc = -4Rrs. \end{aligned}$$

From (16) we obtain:

$$x^3 = -ux^2 - vx - w,$$

that is,

$$x^3 = 2sx^2 - \left(r^2+s^2+4Rr\right)x + 4Rrs. \tag{17}$$

This equation must be satisfied by each of a, b, c, so we get, by substitution:

$$a^3 = 2sa^2 - \left(r^2+s^2+4Rr\right)a + 4Rrs,$$

and two other such equalities in which a is replaced by b and c in turn. By adding these three equalities, we get the following relation (where we have used the short forms $\sum a^2$ for $a^2+b^2+c^2$, and $\sum a$ for $a+b+c$):

$$\begin{aligned} a^3+b^3+c^3 &= 2s \cdot \sum a^2 - \left(r^2+s^2+4Rr\right) \cdot \sum a + 12Rrs \\ &= 4s \cdot \left(s^2 - r^2 - 4Rr\right) - 2s \cdot \left(r^2+s^2+4Rr\right) + 12Rrs \\ &= 2s\left(s^2 - 3r^2 - 6Rr\right). \end{aligned}$$

AVIPSHA NANDI is the daughter of Sri Subrata Kumar Nandi and Smt Ranju Nandi. She is currently studying in Sri Chaitanya Techno School, Bangalore, in grade 9. She has qualified in many state level math Olympiads. She has a keen interest in solving problems in Euclidean geometry using trigonometry. She enjoys drawing and listening to music. She has trained in classical dance and has won many awards. She thanks her parents and teachers for what she has learned. Avipsha may be contacted at avipshanandi05@gmail.com.