feature

Paul Erdős

The Artist of Problem-Posing

Notes from a small suitcase

Paul Erdős has been described as one of the most universally adored mathematicians of all time. No mathematician prior to him or since has had quite the lifestyle he adopted: the peripatetic traveller living out of a suitcase, moving from one friend's house to another for decades at a stretch, and all the while collaboratively generating papers; no one has had quite the social impact he has had, within the community of mathematicians. This article offers a glimpse of his life and work.

R. Ramanujam

A theorem on Facebook

It's very likely that you have a Facebook account, and of course, you have many friends on Facebook. If X is your friend on Facebook, then you are X's friend on Facebook too. But it's possible that Y is X's friend and not yours. Still, I am sure you and X have many common friends, forming a *trio* of friends. Now, here is a question for you.

What is the smallest number n of people on Facebook such that there is ***definitely*** *a trio among them, either all of whom are friends with each other, or none of whom are friends with each other?*

With three people, say A, B and C, it is easy that we will not have this property: let A and B be friends, neither of whom are friends with C. What about four people, A, B, C and D? Again it is easy: make A and B friends, C and D friends, and no more. In both these case the desired trio is not to be found.

When we have five people, it is a little more difficult, but a picture can help think about it. Let us have points denoting the 5 persons; draw a red line connecting them to denote that they are friends on Facebook, and a blue line between them to denote that they are not. Now a red pentagon with the persons on vertices and blue lines to 'opposite' vertices (as in Figure 1) should convince you that we can indeed have a situation without the desired property.

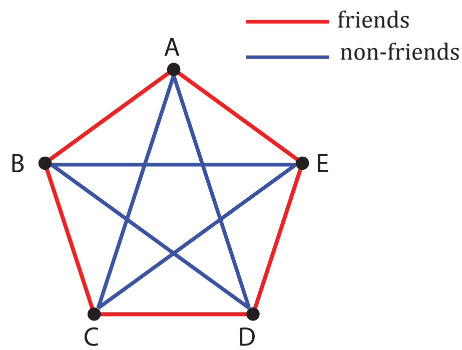

Figure 1: Pentagon with 'red' edges, 'blue' diagonals: no trio of the desired kind

Now to six. Take some time off now to draw pictures like the above. The hexagon with its diagonals does not help; while we get many interesting pictures, we get none that works like the one with five vertices. At this point, we start suspecting that six might indeed be the smallest number we seek. But then we need a proof that among **any six** persons on Facebook, we have a trio, either all of whom are friends, or not-friends.

Call the newcomer F. We first observe that we already know something about F!

Claim. Among the other five persons, there are at least three among them such that F is a friend of all three, or F is friends with none of the three.

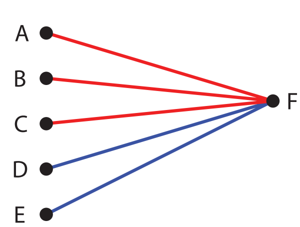

Figure 2: Whom is F friends with?
This figure shows one of many possibilities.

Why? Suppose not. Let us argue with reference to a picture like the one we drew earlier; see Figure 2. We focus on the 'lines' coming out from F. Each line is red or blue. Since there are five such lines, one colour occurs three times or more. Whichever that colour is, we get the three persons we want. (If this colour is red, then the three persons the lines connect to are friends of F, else they are non-friends.)

Nice. Suppose that the three persons identified in the previous step are *A, B, C*. (It could be any three; we have renamed them as *A, B, C*.) Their relationship with F is the same: all friends or all non-friends. Suppose they are all friends of F. Now if any two of *A, B, C* are friends with each other, these two together with F form a trio of friends. And if no two among *A, B, C* are friends with each other, then *A, B, C* form a trio of non-friends! Either way we get the trio we need.

Please check that all possible cases can be disposed of in a similar way. So we have proved a **Facebook Theorem** that is valid for any six of the millions of members who use that site, knowing nothing at all about them!

The picture that we drew was a *graph*, with *edges* connecting pairs of *vertices*. We used two kinds of edges, red and blue. We can call this an *edge-colouring* of the graph with two colours. When every pair of vertices has an edge between them, we call it a *complete* graph. A complete graph on n vertices is denoted by K_n. (So K_2 is just an edge, K_3 is a triangle, and K_4 is a quadrilateral with its two diagonals.)

In this language, what we showed was the following: if each edge of K_5 is coloured red or blue, then a monochromatic K_3 may not get created, but if each edge of K_6 is coloured red or blue, then a monochromatic K_3 necessarily does get created. ('Monochromatic' means that all edges have the same colour.)

The critical number 6 is an example of a **Ramsey number** (named after the mathematician and logician Frank Plumpton Ramsey) of a graph, the minimum number of vertices needed to force a monochromatic subgraph inside it. More rigorously, given any two numbers s and t, the

Ramsey number $R_2(s, t)$ is the smallest integer m satisfying the property that if the edges of K_m are coloured red or blue, then no matter which way it is done there is either a subgraph K_s with all red edges, or a subgraph K_t with all blue edges. With k colours, we can similarly speak of $R_k(s, t)$. What we showed above was: $R_2(3, 3) = 6$.

Why should anyone care about Ramsey numbers? For one reason, finding them is extremely hard! Only a handful are known, and Table 1 lists all the known Ramsey numbers of the form $R_2(s, t)$. You will find it a nice challenge to show that $R_2(3, 4) = 9$.

In the absence of any practical algorithm for computing exact values of Ramsey numbers, a great deal of research effort has been concentrated on obtaining bounds instead. For *diagonal* Ramsey numbers, i.e., Ramsey numbers of the form $R_2(s, s)$, some bounds are known. For instance, it can be shown without too much difficulty that $R_2(s, s) \leq 4^s$, an upper bound. Getting lower bounds is much harder.

s	3	3	3	3	3	3	4	4
t	4	5	6	7	8	9	4	5
$R_2(s, t)$	9	14	18	23	28	36	18	25

Table 1: All the known Ramsey numbers

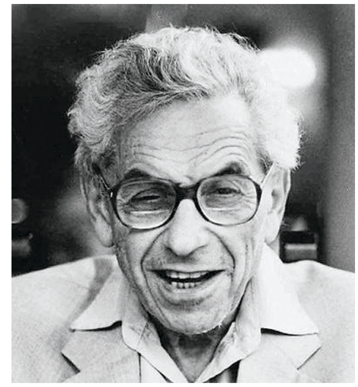

Figure 3: Paul Erdős having a chuckle.
Source: http://24.media.tumblr.com/tumblr_maobc7dXYQ1qipuzxo1_1280.jpg

In 1947, the mathematician **Paul Erdős** (pronounced Air-dsh) proved this remarkable theorem

Theorem 1. *Let k, n be positive integers such that* $2\binom{n}{k} < 2^{\binom{k}{2}}$. *Then* $R_2(k, k)$ *is greater than n.*

In order to show that $R_2(k, k) > n$, it suffices to show that there exists at least one colouring of the edges of K_n which results in no monochromatic K_k. Erdős showed this probabilistically! The details are given in Figure 4.

Erdős's probabilistic proof of Theorem 1.

Consider an edge blue/red colouring of K_n in which the colour for each edge is assigned *randomly* and *independently*, with probability $^1/_2$ for each.

How many copies are there of K_k in this configuration? Clearly as many as there are subsets of size k in the set $\{1, 2, 3, \ldots, n\}$, i.e., $\binom{n}{k}$. What is the probability that any particular copy is monochromatic? Each of the $\binom{k}{2}$ edges in the chosen K_k gets a particular colour with probability $^1/_2$, and there are two colours to choose from, so the probability is equal to

$$2 \cdot \frac{1}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Hence the probability that there exists a monochromatic K_k is at most $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$.

(For, the probability of a union of several events is at most the sum of the probabilities of the individual events.)

This quantity is less than 1 by the assumption of the theorem, hence the probability that there exists a colouring with **no** monochromatic K_k is greater than 0. Therefore, there exists a colouring with no monochromatic K_k, and we are done.

Figure 4: A random proof!

The master who could count by tossing coins

Erdős was 33 years old when he proved Theorem 1. This way of proving the existence of something by showing that the probability that it exists is positive is typically Erdős's. He loved the **probabilistic method** and used it to great advantage to solve problems in number theory, combinatorics and graph theory. He would set up some mechanism for counting the number of ways of doing something, compute the probability of an event, and show that some mathematical object exists. Joel Spencer, who did a lot of work with Erdős refers to this as 'Erdős magic'.

Erdős could ask questions on counting pretty much anything. Consider k points and t lines on the plane. We might ask many questions concerning them, but here is one of Erdős's questions: What is the maximum number $f(k, t)$ of incidences between the points and the lines? He conjectured that the points of a square grid and a certain set of lines give the optimal order of magnitude. This was confirmed only decades later by Szemerédi and Trotter, in 1983.

Here is Erdős proposing a question for the student journal *Quantum*. Let $f(n)$ be the largest integer for which there is a set of n distinct points $x_1, x_2, \ldots x_n$ in the plane such that for every x_i there are at least $f(n)$ points x_j which are equidistant from x_i. Determine $f(n)$ as accurately as possible. Is it true that $f(n)$ is approximately n^ϵ for every $\epsilon > 0$? Erdős offered \$500 for a proof and 'much less' for a counterexample. The question was settled in 1990.

This was also typical of the Erdős style; he posed thousands of problems, and offered prize money for solving many of them.

Very early on, Erdős was attracted to number theory, but there too he turned to counting orders of magnitude. In 1934, when he was 21 years old, Erdős heard of Simon Sidon's work on sequences of integers with pairwise different sums. In 1938 he asked: what is $f(n)$, the maximum number of positive integers $a_i \leq n$ such that the pairwise products $a_i a_j$ are all distinct? He answered the question by reducing it to a question in graph theory.

Through his questions, Erdős led us in many directions that we could not have imagined to exist. Here is an example. In 1927 van der Waerden published a celebrated theorem, which states that if the positive integers are partitioned into finitely many classes, then at least one of these classes contains arbitrarily long arithmetic progressions. In 1936, Erdős and Turàn realised that it ought to be possible to find arithmetic progressions of length k in any 'sufficiently dense' set of integers, which would show that the partitioning in van der Waerden's theorem was, in a sense, a distraction. The conjecture was proved by Szemerèdi in 1974. Not only is it a very difficult proof, but the *regularity lemma* that he used in the proof has become a central tool in graph theory and theoretical computer science. (Szemerèdi was awarded the prestigious Abel prize last year.)

> **A question both deep and profound**
> **Is whether a circle is round.**
> **In a paper of Erdős**
> **Written in Kurdish**
> **A counter example is found.**

Erdős made another related conjecture, far more famous and still open. *Let X be any set of positive integers such that the series* $\Sigma_{x \in X} \frac{1}{x}$ *diverges. Then X contains arbitrarily long arithmetic progressions.* Note that the set of primes is an example of such a set. The general question is open (as noted), but Green and Tao showed in 2004 that the primes contain arbitrarily long arithmetic progressions. (Terence Tao was awarded the Fields prize in 2006.)

All this is very deep mathematics, but what about the fun part? Often, it was recreational mathematics that led Erdős to the deep end. Here is an example. A distinct pair of numbers (*m, n*) is said to be *amicable* if the sum of the proper divisors of *m* is *n*, and vice versa. The smallest such pair is (220, 284). It is still unknown if there are infinitely many amicable pairs, but Erdős showed that the set of amicable numbers has density zero. This means, roughly speaking, that they are quite rare.

The man without boundaries

By now the picture of Paul Erdős, the great problem solver and problem poser, must have taken shape. But he not only posed problems, he also sought out people to pose the problems to. He offered sums of money as encouragement for students to think about problems. Much of this money was his own, he gave freely to numerous non-mathematical charities and causes as well, keeping hardly any money for himself.

Figure 5: Two images of Paul Erdós.
Source: http://www-history.mcs.st-and.ac.uk/Mathematicians/Erdos.html

When he published his first paper in 1932 Erdős was merely 18. He continued to publish until 2003, almost 7 years after his death as some straggling papers continued to be published posthumously! He published 1521 papers in all, collaborating worldwide with a staggering number of mathematicians. How could he do this?

Erdős was prolific because his life was wholly devoted to mathematics. He did not have a job, a regular place of stay, or more possessions than he could carry with him in his two (half empty!) suitcases. He travelled from university to university, from mathematician to mathematician, working until his collaborator was exhausted, and then moving on. He did not cultivate human contact outside of his mathematical interactions, with the exception of his mother, whom he loved dearly. He didn't have to cook, clean or keep house; he had a cadre of people who looked after him, saw to it that he had food, shelter and, when necessary, a visa for his next destination.

Even the language of Paul Erdős was idiosyncratic. To him, children were 'epsilons', people 'died' when they stopped doing mathematics, and people 'left' when they actually died. He didn't lecture, he 'preached', and when he was ready to do mathematics, 'his brain was open'. To him, God was the 'Supreme Fascist'. But there was something that *was* divine for him: he used to speak of **The Book** in which all beautiful theorems and proofs was written down; the job of the mathematician was only to find them. When he found a really elegant argument, he would exclaim, "Ah, that's from The Book!"

Paul Erdős had a hard life. Born in 1913 in Budapest, Hungary, he was a child at the time of World War I, and the years after the war were worse. Jews were not allowed to attend university, and Erdős had to pass a national examination in 1930 before he was exempt from these 'fascist' rules. He attended the University Pazmany Peter in Budapest from 1930 to 1934 and then, fleeing the repressive regime in Hungary, went to Manchester in England for research. His mathematical wanderings began and he worked also in Cambridge, London, Bristol and other places.

By 1938 he could no longer safely return to Hungary because of Hitler's control of Austria, and Erdős spent a year at the Institute for Advanced Study in Princeton University in the USA. After a year, he left Princeton, and started wandering, university to university, mathematician to mathematician, and conference to conference. In 1945 he received word that most of his extended family had been killed in Auschwitz and that his father had died of a heart attack in 1942. He visited Hungary, and spent time in England and the USA. But by 1954, he had problems with the USA which refused to issue an entrance visa for him, alleging communist sympathies. Eventually his entry was eased, after petitions from mathematicians, but all this made Erdős sceptical of nations and boundaries.

Erdős collaborated with 509 authors, nearly twice as many as the next most well connected mathematician. He collaborated so much that the most accepted measure of connectedness is the **Erdős number (EN):** the simple distance connecting a person with Erdős by co-authorship. Erdős himself has EN zero; the lucky 509 co-authors have EN one, and those who have co-written an article with one of this group has EN two (there are 6984 of them), and so on. [1]

If ever there was a mathematician who knew no boundaries, national or subject-wise, it was Paul Erdős, the ultimate problem solver and problem poser.

Further reading

- A. Baker, A. Bollobas and A. Hajnal, *A Tribute to Paul Erdős*, Cambridge University Press, 2012.
- Deborah Helligman, *The Boy Who Loved Math*: *The Improbable Life of Paul Erdős*, Roaring Brook Press, 2013.
- Paul Hoffman, *The man who loved only numbers*, Hyperion, 1999.
- Bruce Schechter, *My Brain is Open: The mathematical journey of Paul Erdős*, Simon and Schuster, 2000.

1. The author of this article is proud to be among the 26,422 who have an Erdős number of three.

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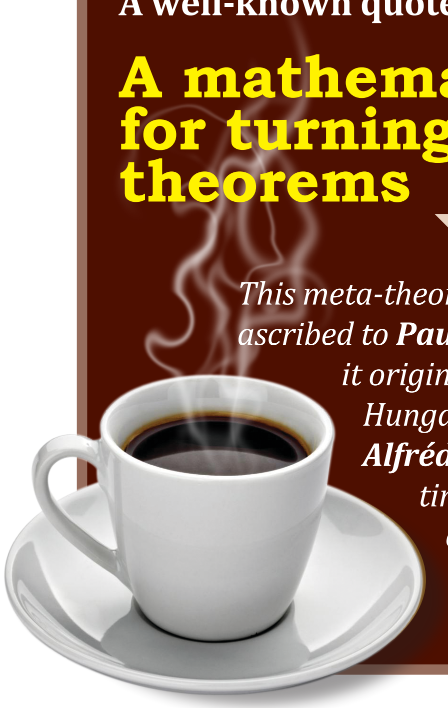
A well-known quote, and a favourite among mathematicians, is:

A mathematician is a machine for turning coffee into theorems



*This meta-theorem has been widely ascribed to **Paul Erdős**, but most likely it originated from another Hungarian mathematician, **Alfréd Rényi**, who was a long-time friend and colleague of Erdős's.*

*A third Hungarian, **Paul Turán**, added the following:*

Weak coffee is suitable only for lemmas.