feature

PHTs …Primitive and beautiful

Harmonic Triples

Part–2

Read how the same simple relationship connects the side and diagonals of a regular heptagon and prove this using the triple angle identities from trigonometry. Then prove the same result using the little known Ptolemy's theorem. And finally, learn how to generate these lesser known triads — Primitive Harmonic Triples.

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In Part I of this article we introduced the notion of a *primitive harmonic triple* ('PHT') as a triple (a, b, c) of coprime positive integers satisfying the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Examples: $(3, 6, 2)$ and $(6, 30, 5)$. We had listed various geometric and physical contexts in which this equation surfaces. We had also mentioned that the equation arises in connection with the diagonals of a regular 7-sided polygon. We start by studying this problem.

Diagonals of a regular heptagon

Given a regular heptagon, one can draw $\binom{7}{2} = 21$ different segments connecting pairs of its vertices. But these 21 segments come in just three different lengths: its diagonals come in two different lengths, and then there is the side of the heptagon.

Let a and b be the lengths of the longer diagonal and the shorter diagonal (respectively), and let c be the side of the heptagon, so $a > b > c$ (see Figure 1); then the claim is that $1/a + 1/b = 1/c$. We provide two proofs for this claim.

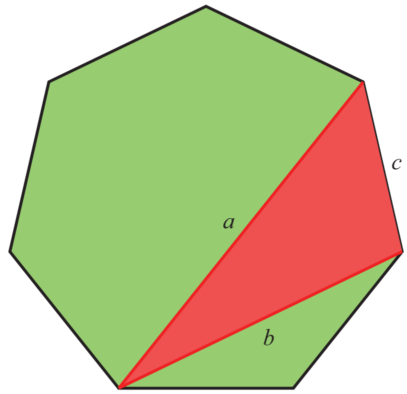

FIGURE 1. Regular heptagon and inscribed triangle; claim: $1/a + 1/b = 1/c$

A trigonometric proof. The angle subtended by each side of a regular heptagon at the centre of the circumscribing circle is $360^\circ/7$. It follows that in the shaded triangle shown in Figure 1, with sides a, b, c, the angles are (respectively) $720^\circ/7$, $360^\circ/7$ and $180^\circ/7$. For convenience let us denote $180^\circ/7$ by θ; then the angles of the triangle are $4\theta, 2\theta, \theta$, and the lengths of the sides opposite these angles are, by the sine rule, proportional to $\sin 4\theta, \sin 2\theta, \sin\theta$ respectively. So the claim that $1/a + 1/b = 1/c$ is equivalent to the claim that if $\theta = 180^\circ/7$, then:

$$\frac{1}{\sin 4\theta} + \frac{1}{\sin 2\theta} = \frac{1}{\sin\theta}, \tag{1}$$

and this is what we now establish. Using the well-known double- and triple-angle identities we rewrite the identity in various equivalent forms:

$$\begin{aligned} \frac{1}{\sin 4\theta} + \frac{1}{\sin 2\theta} = \frac{1}{\sin\theta} \quad &\Leftrightarrow \quad 1 + \frac{\sin 4\theta}{\sin 2\theta} = \frac{\sin 4\theta}{\sin\theta} \\ &\Leftrightarrow \quad 1 + 2\cos 2\theta = 4\cos\theta\left(2\cos^2\theta - 1\right) \\ &\Leftrightarrow \quad 1 + 2\left(2\cos^2\theta - 1\right) = 4\cos\theta\left(2\cos^2\theta - 1\right) \\ &\Leftrightarrow \quad 8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0. \end{aligned}$$

Hence the relation $1/a + 1/b = 1/c$ is equivalent to the following: if $\theta = 180^\circ/7$, then

$$8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0. \tag{2}$$

So if we prove (2) we also prove (1). To prove (2) we note that since $7\theta = 180^\circ$, we have the relation $3\theta = 180^\circ - 4\theta$, and therefore $\sin 3\theta = \sin 4\theta$. This yields, via the double- and triple-angle identities:

$$\begin{aligned} &3\sin\theta - 4\sin^3\theta = 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta\left(2\cos^2\theta - 1\right), \\ &\therefore 3 - 4\sin^2\theta = 4\cos\theta\left(2\cos^2\theta - 1\right) \qquad [\text{since } \sin\theta \neq 0], \\ &\therefore 3 - 4\left(1 - \cos^2\theta\right) = 4\cos\theta\left(2\cos^2\theta - 1\right), \\ &\therefore 8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0. \end{aligned}$$

Thus (2) is established, and hence (1).

A proof using Ptolemy's theorem. There is an elegant proof of the above equality using 'pure geometry', but it requires the use of a theorem which is not so well known at the high school level (though it ought to be better known, as it is such a nice and useful result). The theorem is due to Ptolemy. Here is its statement: *If $PQRS$ is a cyclic quadrilateral, then its sides obey the following equality:* $PQ \cdot RS + PS \cdot QR = PR \cdot QS$. That is, the sum of the products of pairs of opposite sides equals the product of the diagonals. Ptolemy's theorem can be applied in many kinds of settings and yields many nice results. (Of course, we first need to identify a suitable cyclic quadrilateral.) (*Editor's note:* Ptolemy's theorem will be taken up in the 'Geometry corner' of a subsequent issue of *At Right Angles*.)

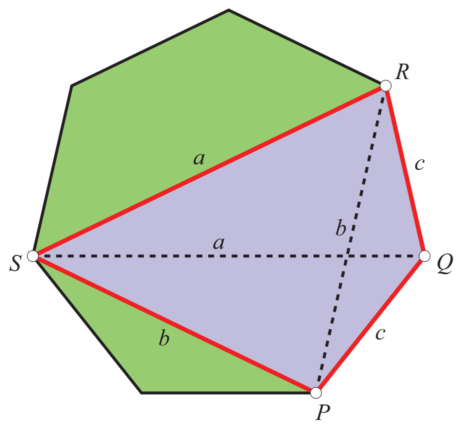

FIGURE 2. Regular heptagon and a cyclic quadrilateral inscribed in it

Here, we simply apply Ptolemy's theorem to cyclic quadrilateral $PQRS$ whose vertices P, Q, R, S are chosen as shown in Figure 2. (The quadrilateral is cyclic since any regular polygon is cyclic, and P, Q, R, S are vertices of a regular heptagon.) Note that PQ and QR are sides of the heptagon (both have length c), RS is a 'long' diagonal (length a), SP is a 'short' diagonal (length b), and its diagonals QS and PR have lengths a and b respectively. Hence by Ptolemy's theorem:

$$bc + ac = ab.$$

Dividing through by abc, we get $1/a + 1/b = 1/c$, as claimed. (Simple, no? But it did require spotting a suitable quadrilateral ....)

Generation of PHTs

Now we take up the question of how to generate primitive harmonic triples in a systematic and mathematically 'nice' way. (So we avoid 'brute force enumeration'.) To avoid listing the same solution more than once (i.e., listing both (a, b, c) and (b, a, c); clearly if one of these is harmonic, so is the other), we shall assume right through that $a \le b$.

At the start we draw attention to a feature about PHTs which makes them different from PPTs. In the case of a Pythagorean triple it is easy to show that if two numbers of the triple are multiples of some number k, then so must be the third number of the triple; e.g., consider the triple $(6, 8, 10)$. Strangely, this property does not hold for harmonic triples! A nice example is the harmonic triple $(10, 15, 6)$; here, each pair of numbers shares a factor exceeding 1, but this factor fails to divide the third number in the triple. The same is true for the PHT $(21, 28, 12)$. (Nevertheless we call such triples 'primitive', because there is no factor common to all the three numbers.)

Systematic generation of harmonic triples. As with PPTs, there are many ways in which we can track down the full family of PHTs. We use an approach based on factorization.

We first clear fractions and get the relation $c(a + b) = ab$. We write this as:

$$ab - ac - bc = 0. \tag{3}$$

If we try to factorize $ab - ac - bc$ we find there is a 'term missing': the expression is 'almost' equal to $(a-c)(b-c)$ but not quite. So we put it in the missing term (which is clearly c^2) and write $ab - ac - bc + c^2 = c^2$. Factorizing this we get:

$$(a-c)(b-c) = c^2. \tag{4}$$

From this we see that $a - c$ and $b - c$ are a pair of *complementary factors* of c^2. (Two factors of a number are called 'complementary factors' if their product equals that number; e.g., 2 and 5 are complementary factors of 10.) Right away we get a method of generating solutions to the harmonic equation — the **'method of complementary factors'**. We express it algorithmically as follows.

(i) Select any positive integer c.

(ii) Write c^2 as a product $u \times v$ of two positive integers, with $u \leq v$.

(iii) Let $a = c + u$ and $b = c + v$.

(iv) Then (a, b, c) is a harmonic triple in which $a \leq b$. To check that it is harmonic:

$$\begin{aligned}\frac{1}{a} + \frac{1}{b} &= \frac{1}{c+u} + \frac{1}{c+v} = \frac{1}{c+u} + \frac{1}{c + c^2/u} \\ &= \frac{1}{c+u} + \frac{u}{c(c+u)} = \frac{c}{c(c+u)} + \frac{u}{c(c+u)} = \frac{c+u}{c(c+u)} = \frac{1}{c}.\end{aligned}$$

The triple may not be primitive, so we must work out how to ensure this. But it is clear that by selecting all possible values of c, and by factorizing c^2 is all possible ways, we will get all possible harmonic triples. Here are two worked examples.

- Let $c = 6$; then $c^2 = 36$. Choose the factorization $c^2 = 2 \times 18$. This yields $a = 6 + 2 = 8$ and $b = 6 + 18 = 24$, and we get the harmonic triple $(8, 24, 6)$. Note that it is not primitive.
- We again let $c = 6$, but change the factorization to $c^2 = 4 \times 9$. Now we get $a = 6 + 4 = 10$ and $b = 6 + 9 = 15$, and we get the harmonic triple $(10, 15, 6)$, which is primitive.
- Let $c = 10$, and choose the factorization $c^2 = 2 \times 50$. This yields $a = 10 + 2 = 12$ and $b = 10 + 50 = 60$. We get the triple $(12, 60, 10)$. Note that it is not primitive.
- Let $c = 10$, and choose the factorization $c^2 = 4 \times 25$. This yields $a = 10 + 4 = 14$ and $b = 10 + 25 = 35$. We get the triple $(14, 35, 10)$, which is primitive.

It appears that *for (a, b, c) to be primitive, we must choose the factorization $c^2 = u \times v$ such that u and v are coprime*. This is so and we take up the proof in Part III of this series. (We pose this as a problem for you, below.) Table 1 gives a list of a few primitive harmonic triples generated this way.

$(2, 2, 1)$,	$(3, 6, 2)$,	$(4, 12, 3)$,	$(5, 20, 4)$,
$(6, 30, 5)$,	$(7, 42, 6)$,	$(8, 56, 7)$,	$(9, 72, 8)$,
$(10, 15, 6)$,	$(10, 90, 9)$,	$(14, 35, 10)$,	$(18, 63, 14)$,
$(21, 28, 12)$,	$(22, 99, 18)$,	$(24, 40, 15)$,	$(30, 70, 21)$,
$(33, 88, 24)$,	$(36, 45, 20)$,	$(44, 77, 28)$,	$(55, 66, 30)$,
$(60, 84, 35)$,	$(65, 104, 40)$,	$(78, 91, 42)$,	$(105, 120, 56)$.

TABLE 1. Some PHTs

Questions to ponder

(1) We stated above that for (a, b, c) to be primitive, we must choose the factorization $c^2 = uv$ in such a way that u and v are coprime. Why should this be so?

(2) In Table 1, note the triples $(2, 2, 1)$, $(3, 6, 2)$, $(5, 20, 4)$, $(6, 30, 5)$, ... Each of these has the same form. Find a formula that generates these PHTs, and show that each such triple is primitive.

(3) Add to the list of triples to Table 1 and study the table carefully. Try to find some interesting features that the triples share. (In Part III — the concluding part — of this series we will explore some properties of PHTs.)

(4) Some PHTs can be 'realized' as triangles; for example, there exist triangles with sides $2, 2, 1$ and $10, 15, 6$ respectively. On the other hand there does not exist a triangle with sides $6, 30, 5$; nor does there exist a triangle with sides $5, 20, 4$. (Reason: Each of these violates the triangle inequality.) What extra condition is needed in the factorization method which will yield a PHT that can be realized as a triangle?

(5) On examining the primitive harmonic triples in Table 1, we notice that the following steps sometimes yield a triple which is harmonic:

(i) Choose any two integers a and c, with $a > c$.

(ii) Let $g = \gcd(a, c)$, and let $b = ac/g^2$.

(iii) Then the triple (a, b, c) is sometimes harmonic.

For example, take $a = 14$, $c = 10$; then $g = \gcd(14, 10) = 2$, so $b = 14 \times 10/4 = 35$. We may verify that the triple $(14, 35, 10)$ is harmonic (indeed, it is a PHT).

On the other hand, take $a = 14$, $c = 12$; then $g = 2$, so $b = 14 \times 12/4 = 42$. But the triple $(14, 42, 12)$ is not harmonic.

When do these steps yield a harmonic triple? Obtain a complete answer.

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