feature

Mathematical make-believe?

Mat(c)h made in Heaven

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The Nobel Prize presentation ceremony recently concluded in Oslo, Norway, and like most others, I only managed to read the first two paragraphs of any article on the achievements of these men and women. Being a student of mathematics, and due to the absence of a Nobel prize in that field (rumoured to be due to a disagreement between Alfred Nobel and mathematician Mittag Leffler), I find myself drowning in the technical jargon present in all such write ups. I was thus circumspect when I saw a notice announcing a talk – requiring no prior knowledge of the subject – on the Nobel Prize winning work of Alvin Roth (Economist) and Lloyd Shapley (Mathematician/ Economist).

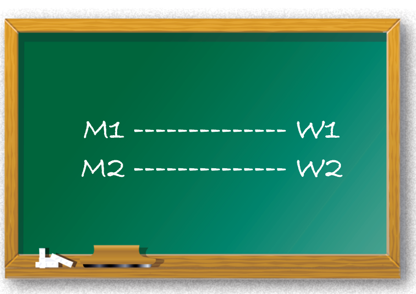
Walking into the packed hall, I went to the back of the room and seated myself so as to be able to make a quiet exit in case I disagreed with the notice on what "no prior knowledge required" meant. In walked our speaker for the day (henceforth referred to as Professor), looking pleased at the large turnout. Setting his notes down on the table he addressed the crowd of eager faces.

"As I mentioned in my mail inviting you all here, I am not an expert on this subject. In fact, this talk is more to celebrate the unprecedented event that I actually understood the work of some economists and I would like to share my excitement with you. Feel free to ask me as many questions as you wish and in turn, allow me the freedom to not know the answer at times!"

I chuckled quietly to myself, pleased at the informal beginning to the proceedings.

"So," he began again, "today we're going to discuss the work of Roth and Shapley. They were just recently awarded the Nobel Prize in Economics for their work on the matching problem. The basic problem, initially worked on by Shapley (along with a mathematician by the name of David Gale) is as follows. Suppose you have n men and n women in a room, each man has a list which rates each of the n women according to who he likes more and similarly, each woman has a corresponding list of men. Suppose a marriage consists of pairing up a man and a woman, i.e., to each man a unique woman is associated and vice versa. For those of you with a mathematics background, a bijection (one to one, onto correspondence) is set up between the men and the women. You are allowed to divorce your spouse if you prefer the husband (correspondingly wife) of another person to your current partner and that person also prefers you to their wife (correspondingly husband)."

He probably sensed our brains furiously trying to wrap itself around the idea, because he soon picked up a chalk and wrote on the board:



"M1 can divorce W1 only if W2 is higher than W1 on his preference list AND W2 prefers him to M2. Now, the question is, is there a marriage arrangement such that all n men and women are matched, and no one wants (or in this case, is permitted) a divorce!"

"Such an arrangement IS possible, and not just that, there is an algorithm by which you can get this 'stable' arrangement, but we'll come to that in a moment. Let me first give you a slightly different example, and one where a stable arrangement is NOT possible. This is the *roommate problem*."

"Suppose you have four people A, B, C, and D who have to share two rooms (two in each room). Again, they all have their own preference lists and the conditions by which you can change rooms is analogous to the divorce scenario in the previous example. Now take these as your preference lists and work out that a stable arrangement is not possible and tell me what the difference between the two examples is."

I whipped out my notebook and began to scribble furiously on the last page, determined not to lose track of things.

Person	Preference
A	B > C > D
B	C > A > D
C	A > B > D
D	C > A > B

I wrote out the various possibilities:

Possible pairs in Room 1	Therefore pairs in Room 2	Preferences Room 1	Preferences Room 2
A & B	C & D	A is happy, B prefers C	C prefers B, D is happy
A &C	B & D		
A &D	B & C		

I thought to myself, "In 1), B and C will be better suited, in 2), A and B will want to room together and in 3), A and C will want to share. So there is no stable arrangement! But what is the difference between this and the marriage problem?!"

"Anyone figured out the difference yet?" asked the Professor.

To my dismay, someone's hand shot up. "Here A can choose from B, C, D whereas in the earlier problem, the men can only choose from the woman and vice versa. It's a modern day marriage problem sir!" he quipped.

Happy with the participation of the audience, the Professor replied smiling, "That's right! In the Marriage Problem, the men ONLY rank the women, and the women can ONLY choose from the men, which is not the situation in the roommate conundrum! Okay, so now that we've established that it isn't a trivial problem that we're attempting to understand, let's think about this algorithm that our economist friends have come up with."

The chalk reappeared in his hand and he began to write again. I fidgeted around, trying to find the optimum angle to look at the board from, kicking myself for my seating choice. I managed to find a position just as he finished his visit to the board.

"Suppose there are 3 men and 3 women, for simplicity's sake," he said, now walking up and down the length of the board, all the while looking at his audience. "Suppose that each man proposes to his favourite lady, and each lady considers all the proposals she receives (possibly none), scrutinizes them and keeps the one which is highest on HER list and rejects the rest. Note that she does not say 'Yes' to the one she keeps, she just tells him 'you're in contention, but hold your horses, I may change my mind yet'. Now all the men who are depressed at the outright rejection get another chance, and they propose to their second favourite woman, and the same procedure repeats itself."

He paused as if for dramatic effect before exclaiming, "This simple technique is the algorithm!"

There was a murmur around the audience as everyone spoke to those sitting beside them, looking slightly comical. Excited and serious is not an expression that the human face has learnt to master!

A faint voice from the back of the room slowly piped up… mine. "Sir, I can see that it seems to give a stable arrangement *(Refer to Box I)*, but how are we guaranteed that this procedure will ever end, and even if it does, it need not be unique, right?"

"Good question! To answer the first part, take this example and try and work it out for yourself and see that there is nothing that you have done that is specific to this example.

Man	Preference
M1	W1>W3>W2
M2	W1>W3>W2
M3	W3>W2>W1

Woman	Preference
W1	M2>M3>M1
W2	M3>M1>M2
W3	M2>M1>M3

(Answer on Page 30)

Here is my reasoning on why the suggested algorithm yields a stable arrangement:

Suppose M (for man) is not married to W (for woman) but yet prefers her to his own wife. We show that W cannot prefer M to her husband. Since M likes W more than his wife, at some point during the algorithm, M would have proposed to W. Since M and W are not together, that means that W rejected M's proposal in favour of someone she liked more! Thus, W must like her current husband more than she liked M and hence there is no instability in our arrangement!

Box I: Reasoning for stability of the arrangement

A more detailed yet easily understood explanation is available in the American Mathematical Monthly where D Gale and L S Shapley published their work."*(Refer to Box II.)*

As mathematicians, however, we will never be satisfied with a proof just because it seems to work in a particular example. On getting back to my room after the talk, I looked up the original paper by Roth and Shapley in the American Mathematical Monthly. The argument is simple. First of all, note that eventually (in fact in $n^2 - 2n + 2$ stages), every girl must have received a proposal.

Suppose some girl hasn't received a proposal. Then, since the number of boys and girls are the same, there must be at least one girl who at that point has at least two proposals. Thus, she must reject all but one and the rejected (and dejected) boys must now propose again. Since no boy can propose to the same girl more than once, every girl MUST receive a proposal sooner or later! And once the last girl receives a proposal, the period for "courtship" is over and the procedure must end and each girl must accept the boy she has on her string!

Box II: Why the procedure will end!

"As for your second question, this solution may not be unique! Suppose that in some parallel universe the women are the ones proposing and the men accepting/ rejecting. Then, using the same algorithm we'll get another marriage arrangement that might well be different. In fact, if you notice, the 'proposing party' gets a favourable result as opposed to the other! This is because the 'proposing party' is in fact going out there in order to look for the best possible deal for them while the 'accepting party' is waiting to decide from the offers they receive! I'm sure there is a life lesson here somewhere but I'm not here to lecture on philosophy!"

The room burst into laughter, mostly at the joke, but partly to express their happiness at having understood the lecture thus far.

"Many years later, Dr Roth (along with a number of fellow academics) modified the Gale Shapley algorithm in different instances and applied the work to a number of areas such as matching hospitals and medical students. Just about ten years or so back, he was asked to sort out the chaotic New York City High School application system. As is the case with any great piece of work, a number of other people have taken up this idea and tried to make it even more streamlined. One such case involves the example of hospital and medical students. A possible aim is to find a system in which if students lie about their preferences, it may not yield a solution in their favour *(Refer to Box III).* Another could be accommodating for married students wanting to be in the same hospital (or town) as their respective spouses. It is a case of taking the above algorithm and making it more attuned to the eccentricities of the real world."

I found myself quite excited, not least because I finally had somewhat of a tangible answer to people asking me what it was I could do after learning so much maths. My customary "the world is at my feet" sort of answer was getting stale to my ears!

"So that was what I wanted to discuss regarding 'the marriage problem'. Now, if nobody minds, there is another, unrelated but all the same interesting topic that I would like to discuss. How are we doing on time?"

I glanced at my watch and found that while we were trying to play match maker, nearly an hour had passed! Expectedly, everyone vociferously nodded their assent and we continued.

"Okay, we now discuss a slightly different problem. Suppose you have a particular town, and in that town a finite number of schools. Let this sheet of paper represent the town," he says holding a colourful sheet; "let the dots from which the colours are radiating be the schools, and let each point on the sheet be a child. Yes, I know that's unrealistic but just bear with me. Each school has a fixed capacity. The question now becomes, how does one allocate students to schools? The natural idea is to use distance:

students who are closer are preferred to those further away. You may object and say that a student may be equidistant from two schools (or vice versa) but this would form what is called 'a set of measure zero'."

I tried to recall my course in measure theory, one semester ago suddenly felt like an eternity away. "Measure zero essentially means of negligible size, like the integers as a subset of real numbers, or, for the poetically inclined, the stars in the sky," I remember my teacher saying and quickly brought my attention back to the lecture.

"So what's happening in this picture is, each school branches out radially until it fills up its quota, and if two school 'kingdoms' touch, then neither of them moves any further in that direction but continue to expand in other directions. Analogous to the divorce concept in the previous problem, a child X may change schools if there is a school A that:

a. Is closer to him than his present establishment, B

b. Has a student Y who is further away from him

Then, much to the displeasure of Y and his parents, he will be asked to leave A, and X will be enrolled in his place. The question then becomes, is there a stable arrangement for this question? The answer to this too is Yes, and, anticipating your next question, this arrangement will be unique! The reason is that, unlike in the marriage scenario, here the criteria for the 'preference lists' is the same for both students and schools! So if men and women find a parameter and an unambiguous rating system on which everyone agrees, then we can find a unique stable arrangement that is at the same time the best and worst possible for both parties!"

Someone seated in the front row asks, "But sir, in this particular diagram, if you notice, there are disjointed bits for some colours. Is it possible for every colour to be 'connected' in some sense?"

"Definitely! This is the next logical question to wonder about and that is exactly what researchers wondered. I won't go into the details of this; perhaps we can have another seminar sometime on this question where we can discuss this question at length. On this colourful note I will end today's lecture. I hope you enjoyed yourself. For those of you who thought that this was a waste of your time, hopefully the samosas and tea outside will make it feel a little more worthwhile!"

Spontaneous applause broke out in the room and everyone was on his or her respective feet, some eager to get their hands on the promised samosa, but more in appreciation of a well-delivered and more importantly, reasonably well-understood lecture – a far from common event in the world of academia!

As I stood waiting for my share of the refreshments, I heard a remark, "If this Ph D thing doesn't work out, maybe I can use this algorithm to open my matrimonial site!" While that, I'm quite certain, wasn't the aim of Messrs. Roth and Shapley, I too realised that Maths, or any subject for that matter, is far more interesting when you look at the problem at hand in a broader perspective, rather than being caught up with whether the expression in line 23 should have a minus sign or not.

(Answer: M2-W1, M1-W3 and M3-W2 will live happily ever after!)

References

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2. D Gale and L S Shapley, College Admissions and the Stability of Marriage, *American Mathematical Monthly* **69**, 9-14, 1962.
3. http://www.nobelprize.org/nobel_prizes/economics/laureates/2012/advanced-economicsciences2012.pdf
4. http://en.wikipedia.org/wiki/Nash_equilibrium

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One particular such “real world eccentricity” is people wanting to subvert the system to obtain results in their favour! That is to say, suppose one man knew the preference list of all the others, would changing his own preference list skew the final arrangement in his favour? The answer to this question is far from obvious, but it is yes! In fact, Roth proved that there is no stable arrangement for which telling the truth is the best strategy for all parties concerned! Let us take the following case as an example to see how one may influence the final arrangement in their favour.

Men	**True Preference**	**Women**	**True Preference**
M1	W1>W2>W3>W4	W1	M4>M3>M2>M1
M2	W1>W3>W2>W4	W2	M4>M1>M3>M2
M3	W1>W2>W4>W3	W3	M1>M2>M4>M3
M4	W3>W4>W2>W1	W4	M2>M1>M4>M3

An easy verification shows that the Gale Shapley Algorithm will now yield the following stable result (under the Women propose scenario):

W2-M4, W3-M1, W4-M2, W1-M3

Now, suppose M4 is not the righteous man we believe him to be and he decides to try to subvert the system. Armed with the knowledge of the preference lists, M4 cunningly changes his list to: W3>W4>W1>W2.

Men	**New Preference**	**Women**	**True (=New) Preference**
M1	W1>W2>W3>W4	W1	M4>M3>M2>M1
M2	W1>W3>W2>W4	W2	M4>M1>M3>M2
M3	W1>W2>W4>W3	W3	M1>M2>M4>M3
M4	W3>W4>W1>W2	W4	M2>M1>M4>M3

The Gale Shapley Algorithm now yields the arrangement given by:

W2–M1, W3–M2, W4–M4, W1–M3

Note now, that as compared to the previous arrangement, M4 is now married to W4 as opposed to W2. Since W4 is higher on his ‘true’ preference list, he has achieved a more favourable result by giving a different rating list!

The reason that our algorithm is still viable in the real world, despite the large number of miscreants trying to find loopholes in the system, is that the volume of information required to foresee the possibilities, and to find a way around it is enormous! Consider the High School application system mentioned earlier. For an applicant to subvert the system, they need to know the preference list of the applicants and (potentially) that of the schools as well. Roth and Rothblum proved that provided the information available to applicants is sufficiently limited, he or she cannot gain by submitting a list which reverses the true ordering of two schools (as M4 did earlier).

In non-cooperative game theory, such a situation (one in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy unilaterally) is known as Nash Equilibrium.

Box III