in the classroom

How To . . .

Solve a Geometry Problem – I

A Three Step Guide

An informal, short guide on solving geometry problems. Ajit Athle describes some strategies which help in solving geometry problems and demonstrates how these strategies are used in solving two intriguing problems.

Ajit Athle

Problem solving in geometry poses special difficulties. Unlike problems in arithmetic or algebra, where one simply starts 'at one end' and proceeds to the 'other end' in a smooth, linear manner, the solution of a geometry problem often gives rise to a blank feeling. One does not know where to start! Often the solution requires the drawing of auxiliary lines and angles, and the figure itself gives no hints. Additionally, one is faced with the task of spotting relationships between pairs of triangles, or pairs of angles, disentangling them from a maze of lines and shapes, and to do so needs a keen eye indeed. This is a skill which can be difficult to cultivate. In short, problem solving in geometry is tough!

In this many-part article we share some thoughts on how to approach this challenging task.

1. Two problems

We first list two problems and invite the reader to spend time on them before reading on.

(1) Outside a given triangle ABC we have a point D (Figure 1) such that $AB = BD = DA$, $\angle ACD = 10^\circ$, $\angle DAC = x^\circ$, and $\angle DBC = (x + 30)^\circ$. Problem: Find x.

(2) Figure 2 shows a circle with a chord AC; the midpoint of arc AC is D. Let B be any point on arc AC such that arc $AB >$ arc BC, and let DE be drawn perpendicular to AB, as shown. Prove that $AE = EB + BC$. (This is the famous *Broken Chord Theorem*, discovered and proved by Archimedes.)

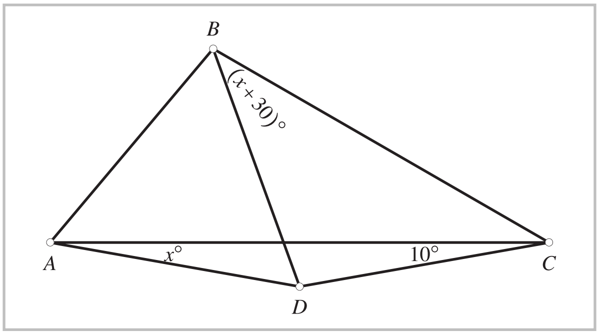

Figure 1

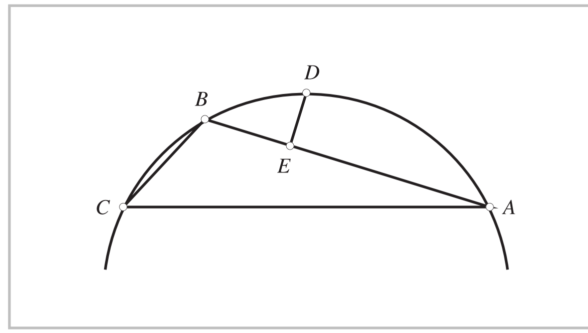

Figure 2

Solutions to the problems

Our approach in general should be to:
(i) Understand the problem well, and record the given information accurately in a (reasonably large) diagram which is a copy of the given figure, making a note of what is required to be determined or proved; (ii) ascertain whether there is any hidden information in the problem statement, and if so, to note this too in the diagram; (iii) draw conclusions which lead us to finding the solution to the problem.

Solution to Problem 1. Let us apply the principles listed above to the problem at hand. Here we have, $AB = BD = DA$ (see Figure 3). What does this imply? Well that is clear: $\triangle ABD$ is equilateral, so $\angle BAD = 60^\circ$. Hence: $\angle BAC = (60 - x)^\circ$ and:

$$\angle ABC = (60 + 30 + x)^\circ = (90 + x)^\circ,$$

$$\angle BCA = 180^\circ - (60 - x)^\circ - (90 + x)^\circ = 30^\circ.$$

We see that $\angle ADB$ is twice $\angle ACB$. Recalling the result that the angle subtended by the chord of a circle at the centre is twice the angle subtended at the circumference, we deduce that if a circle is drawn with D as centre, passing through A and B, then the circle passes through C as well.

This means that D is the centre of a circle passing through points A, B, C. Hence $DA = DC$, both being radii of the circumcircle, and so $\triangle DAC$ is isosceles. It follows that $x = 10$.

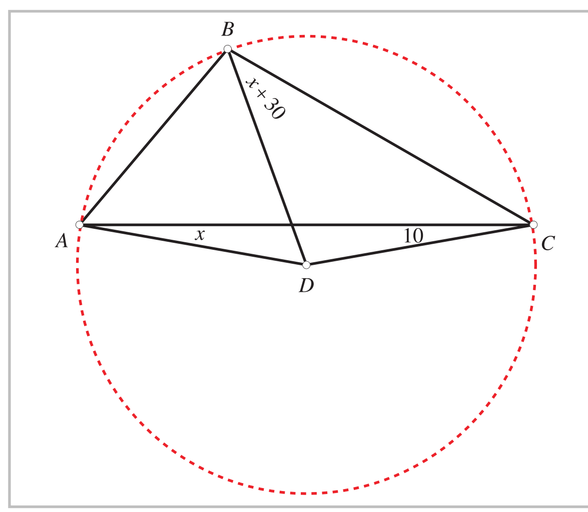

Figure 3

Thus the answer was arrived at in six simple steps using theorems or properties known to all school children. There is much beauty in this simplicity.

In a problem like this, one would try and figure out all angles in the diagram and then see if a special relationships can be observed which lead us to some useful inference. One would not know in advance if any particular angle will be more useful than any other. But that would vary from problem to problem, would it not?

Solution to Problem 2.
(Broken Chord Theorem). In Figure 4, AC is a chord of a circle; D is the midpoint of arc AC; B is a point on arc AC such that arc $AB >$ arc BC; $DE \perp AB$. We must prove that $AE = EB + BC$.

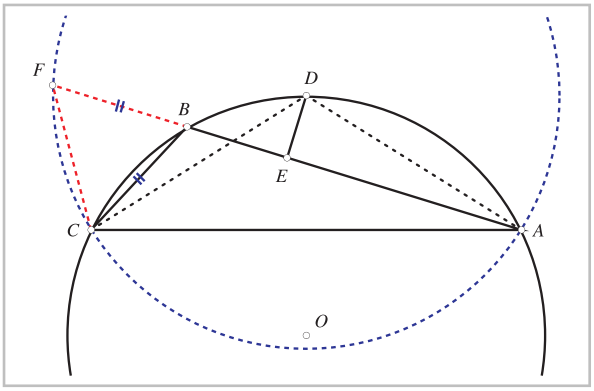

Figure 4

As the segments EB and BC are not in the same line, the problem suggests a natural construction: *Extend AB to F such that* $BF = BC$. We now need to prove that $AE = EF$; i.e., that E is the midpoint of AF.

Let $\angle BFC = x$; then $\angle BCF = x$ since $BC = BF$ by our construction. Hence $\angle CBA = 2x$, implying that $\angle CDA = 2x$ (angles in the same segment). Next, note that D lies on the perpendicular bisector of AC (since D is the midpoint of arc AC), and that AC subtends an angle of $2x$ at D while it subtends an angle of x at F. We infer that D is the circumcentre of the circle passing through points A, C, F. And since DE is perpendicular to AF, it follows that E is the midpoint of AF, which is what we had set out to prove after our construction.

Note how the construction suggests itself and how easy the proof is once the auxiliary lines are drawn.

Closing remarks

'Problem Solving' means engaging in a task for which the solution method is not known in advance. In order to find a solution, one must draw on one's knowledge, and through this process, one develops new mathematical understanding. Solving problems is not only a goal of learning mathematics but also a major means of doing so. When one arrives at the correct solution there is naturally a great deal of satisfaction and sense of self-confidence which gets generated. And that, surely, is one of the things that any teacher is trying to inculcate in a student.

We shall present and solve more such problems in future editions of this column.

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