in the classroom

Pitfalls in ...

The Method of Induction

The perils of teaching by example

Constructive teaching encourages students to recognize patterns and build appropriate theory with its roots in conjecture. What are the dangers in this method? What precautions should the teacher take to avoid conveying the impression that a result is true simply because it has been observed in all the examples considered?

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Inductive and deductive methods are emphasized in the teacher education (B.Ed.) curriculum, in 'methods of teaching mathematics'. I taught the method of induction passionately during my stint as a teacher educator and witnessed many lessons of mathematics taught by student teachers. It was employed by our student teachers whenever they dealt with generalizations and, on occasion, it led to the development of formulae.

How does the 'method of induction' work? Typically a teacher provides specific instances one by one, asks children to observe, note the pattern and extend this pattern to unknown cases, which leads to a generalization. For example, if the teacher were teaching the laws of indices, she would take up the following (or similar) examples, repeatedly ask questions, elicit responses from the students, collate responses on the black board systematically, and thus arrive at the law:

2^3 2^4	2 2 2 2 2 2 2	2^7	2^{3+4}
3^4 3^5	3 3 3 3 3 3 3 3 3	3^9	3^{4+5}
0.5^3 0.5^2	0.5 0.5 0.5 0.5 0.5	0.5^5	0.5^{3+2}
a^4 ^{2}	a a a a a a	a^6	a^{4+2}
y^5 y^2		y^7	
b^m b^n			

This is the crux of the method. At first sight, this looks like a great way of helping children explore generalization in mathematics. Just by working through a few specific examples, the children are able to generalize without the teacher explaining the law! The students discover the rule!

While working as a teacher educator, my focus remained only on the variety of examples the teacher gave the students to examine and the space s/he created for students to look for patterns, hypothesize and arrive at a general form by systematic questioning. I enjoyed watching students come up with hypotheses and I was blissfully unaware of the boundaries that underlie the method.

A few years down the line, as I started thinking about the method, doubts started emerging in my mind. Is this really a discovery by the student? Is there not a need for proof even after we arrive at generalization? Is this proof at all? This article brings together my thoughts on the topic.

What happens in induction?

In general, in induction, we examine a number of particular cases and based on the observations we arrive at a 'generalisation'. If we see something that works several times in a row, we're convinced that it works forever. For example:

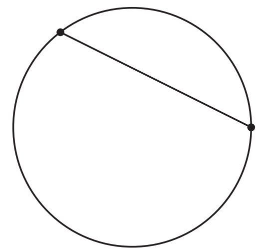
- The cow has four legs.
- The horse has four legs.
- The dog has four legs.
- The elephant has four legs .

Conclusion: "All animals have four legs".

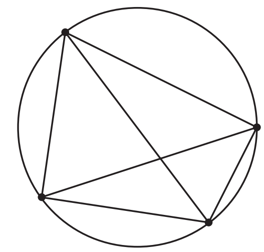
As we see in the above example, induction gets strengthened as the confirming instances pile up. We look at specific examples and conclude it is true for all cases. Do we foresee a problem when this is applied to Mathematics? What is the nature of the kind of generalisation we arrive at?

Is it a proof or conjecture?

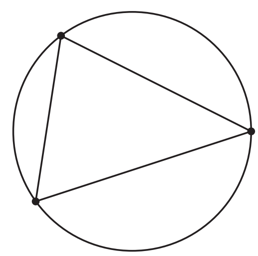
Consider a circle with n points on it. How many regions will the circle be divided into, if each pair of points is connected with a chord? (Assume that no three chords meet in a point.)



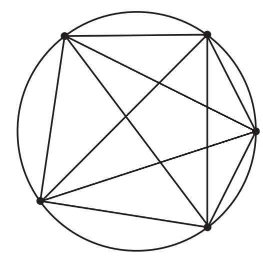
2 points
2 regions = 2^1



4 points
8 regions = 2^3



3 points
4 regions = 2^2



5 points
16 regions = 2^4

By looking at a few examples, most of us would be convinced that with 6 points there would be 32 regions; but this has not been proved, only conjectured. Our guess is that the number of regions when n points are connected is 2n -1. We looked at a few examples, found it true for those specific instances and now believe it works for all unexamined cases. There is no logic to explain why we believe it that way other than "it is true for the cases we have verified". It remains at the level of a proposition that is unproven but thought to be proved. It should be clear that the generalization we arrive at by looking at some examples is only conjectured and not proved.

How do we prove the conjecture?

To prove the conjecture we need to examine all possible cases. For instance, one might conjecture that 1+3+5. . . . . . + (2n-1) = n2 for all natural numbers n. Let us see how we might prove it:

- For n = 1 we check that it is true.
- For n = 2 we check that it is true.
- For n = 3 we check that it is true, and so on.

Of course one easily verifies that the statement is true for the first few (even the first few hundred or thousand) values of n. Yet we cannot conclude that the statement is true unless every possible case is examined. Maybe it will fail at some value not yet checked, who knows? It is not possible to verify the statement for all values of n since there are infinitely many values.

We are quite sure that every even number is divisible by 2. That doesn't mean we have examined every even number, and nobody else has done so either. There is every possibility of finding some big number which may break this rule! How do we ascertain there is no such number? Why isn't the situation like 'all animals have four legs'? In both cases, aren't we moving from a few known cases to unknown cases?

To conjecture that a statement is true merely from empirical evidence is risky. Inductive arguments have premises that purport to support the conclusion; but there is no guarantee of the accuracy of the conclusion. It is insufficient to argue that a mathematical statement is true by simply doing experiments and making observations.

We realize that finding the number of regions when there are six points on the circle does not prove the conjecture. If there are indeed 32 regions, all we have done is shown an instance in support of our conjecture. If there aren't 32 regions, then we have proved the conjecture wrong. (In fact, if we go ahead and experiment, we find that there aren't 32 regions!)

Another example: we may conjecture that $n^2 - n + 41$ is prime for all natural numbers n:

- When $n = 1$, then $n^2 - n + 41 = 41$ is prime;
- When $n = 2$, $n^2 - n + 41 = 43$ is prime and so on.

Even if one continues the experiment till n = 30 one would not find a counter example. But it is easy to see that the statement must be wrong, for when n=41, the expression equals 412 which is clearly not prime. So by finding a counter example, we have proved the conjecture wrong.

You can never prove a conjecture is true by example. But you can prove a conjecture is false by finding a counter-example.

The Need for Proof

While evidence is insufficient to guarantee the truthfulness of the statement, a counter example can be enough to disprove it. But if the statement is true then one may not find a counter example at all!

For instance Fermat (1601-1665) conjectured that when is an integer greater than 2, the equation $x^n + y^n = zn$ admits no solution in positive integers. Attempts by mathematicians in finding a counter example ended in failure. Despite that, we cannot conclude that Fermat's conjecture is true. Hence experimentation with a few examples and the lack of a counterexample to disprove the hypothesis fail as a method of proof in mathematics. Then how can we verify the statement? Is there a way out? A powerful tool is mathematical induction.

Proof by Mathematical Induction

Mathematics distinguishes itself from other disciplines in its structure and its internal consistency. It is built upon axioms and postulates, which are self-evident truths and thus accepted without proof. All theorems, principles and generalisations in mathematics are derived and proved based on these. Thus it is with the method of mathematical induction. There is a clear cut distinction between the two cases; 'animals having 4 legs' and 'even numbers divisible by 2'. Even numbers are defined by a definite rule. We prove the smallest even number (2) is divisible by 2. This forms our 'small sample'. Using the rule we prove that the next even number after every even number divisible by 2 will also be divisible by 2. This is our rule for unexamined cases. This is enough to imply that the successor of 2, i.e. 4, is divisible by 2; so also 6 (the successor of 4); 8 (the successor of 6); and so on, ad infinitum. This is how a small sample and a rule about unexamined cases can give us information about every case. This is how our knowledge of an infinite set of unexamined cases can be as certain as the conclusion of a valid deduction, unlike the conclusion of an ordinary induction. The definition of even number contains this property within itself!

The large-scale structure of a proof by mathematical induction is simple:

1. Prove the theorem for the known cases.
2. Prove that if the theorem is assumed true for any value of n, then it is true for the next value of n.
3. Connect steps 1 and 2 and deduce that since the theorem is true for the known case (say n =1), it will be true for the next case (n=2). Similarly, it will be true for n=3 and so on, for all finite positive integers n.

Induction and Deduction

"Mathematical induction" is unfortunately named, for it is unambiguously a form of deduction! However, it has similarities to induction which likely inspired its name. It is like induction, in that it generalizes to an infinite class from a small sample. Mathematical induction is deductive, however, because the sample together with a rule about the unexamined cases actually gives us information about every member of the class. So the conclusion of a mathematical induction does not contain more information than was contained in the premises, but it concludes with deductive certainty. (Peter Suber 1997).

Conclusion

In mathematics, as in science, there are two methods by which we can arrive at new results. One, deduction, involves assuming a set of axioms from which we deduce other statements, called theorems, according to prescribed rules of logic. This is the method used in Euclidean geometry.

The second method, induction, involves guessing general patterns from observed data. In most branches of science, such guesses remain merely conjectures, with varying degrees of probability of correctness. In mathematics, however, certain conjectures can be proved by the technique called 'mathematical induction'. This technique is not 'induction' in the usual sense of the word; rather, it is a method for proving conjectures that have been arrived at by induction.

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