math club

# find the next number!

in the classroom

Sequences are an excellent topic for a math club, in part because they offer something for everyone. And since there is no dearth of suitable sequences, we will never run out of material. Perfect for a Math Club!

Here we study a problem which involves drawing lines on a sheet of paper. All we do is draw straight lines using a pencil and count regions! The problem may be posed in an imaginative way: *If we make 10 straight cuts through a flat pizza using a knife, how many pieces can we get?* Here 'pizza' can be replaced by any flat object (a sheet of paper, a pancake, an omelette, a *dosa*). The only rule to be followed is that while the cuts are being made, the pieces are not rearranged in any way (for example, we cannot stack the pieces on top of each other before making the next cut); we simply run the knife through the object 10 times in succession. Ambiguities in meaning can be avoided by dealing with a plane sheet of paper and a pencil: *If we draw 10 straight lines on a sheet of paper, what is the largest number of regions we can get on the paper?* Naturally, the '10' does not have any significance; it can be replaced by any other number. So the mathematical essence of the problem is this: *If we draw n straight lines upon a sheet of paper, what is the largest number of regions we can create on the paper?* Here $n$ represents an arbitrary positive integer. If we let $R(n)$ denote the number of regions created by $n$ lines, we want a formula for $R(n)$. (See Figure 1.)

Why 'largest'? Well, we can 'lose' regions if we are not careful. We could draw two lines parallel to each other and so lose regions; or we could draw several lines through the same point and so lose regions (see Figure 2; in (a) we get only 3 regions when we could have got 4, and in (b) we get 6 regions when we could have got 7).

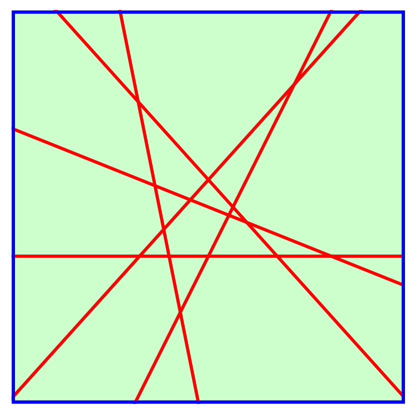
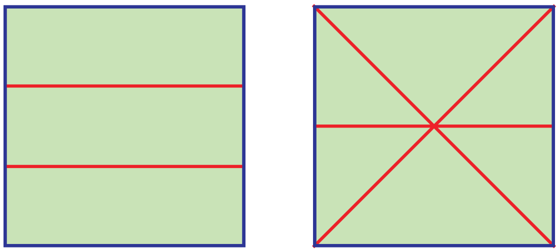


Figure 1 The picture for $n = 6$ is bad enough!

We shall assume that we do not let either of these situations occur. That is, we shall draw the lines in such a way that (i) every line meets every other line somewhere on the sheet, and (ii) no three lines pass through a single point. Given these two conditions, can we find a formula for the number of regions created by $n$ lines? A related question is the following: *Can it happen that the above two conditions are satisfied, but there are different configurations possible which give different numbers of regions?* This is far from obvious! The first few values of $R$ are easily found: $R(1) = 2$, $R(2) = 4$, $R(3) = 7$, $R(4) = 11$. For $n = 4$ we need to draw some trial configurations



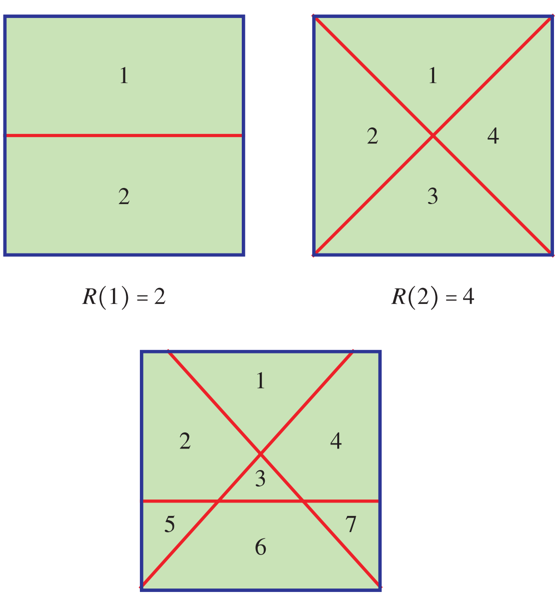
(a) (b)

Figure 2 'Losing' regions, in two different ways

before we are convinced that we have the right answer (see Figures 3 and 4).

Examining these values, we quickly spot a pattern (in a Club setting, we will of course coax the children to do the spotting): we see that $4-2=2$, $7-4=3$, $11-7=4$. *The differences between successive entries appear to advance by* 1 *each time.* If this pattern persists then we expect that $R(5)-R(4)=5$, and hence that $R(5)=16$. Is $R(5)$ really equal to $11+5=16$? Is $R(6)$ really equal to $16+6=22$? And does the difference pattern continue? Experiment and find out!

We leave the matter open here, and end by posing a few questions for further exploration.




$R(1)=2$ $R(2)=4$

$R(3)=7$

Figure 3 The cases $n=1$, $n=2$ and $n=3$

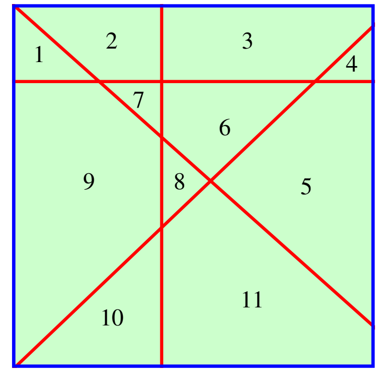



Figure 4 $R(4)=11$ (Please check that we cannot get more than 11 regions)

## Some questions to ponder . . .

(1) Assuming that the pattern described above continues, what do we expect will be the value of $R(10)$? $R(20)$?

(2) Assuming that the pattern described above continues, can we find a simple formula for $R(n)$?

(3) We have repeatedly used the phrase " . . . assuming that the pattern persists . . . ". But why *should* the pattern persist? What geometric logic can we give for believing that it will persist? Unless we can give a convincing answer for this, our answer for question #2 will at best be partial and therefore of limited value.

(4) Could there be some other way of proving the formula we find for $R(n)$?

(5) The sequence $R(1), R(2), R(3), R(4), \ldots$ is defined *geometrically*, but it may have some interesting *arithmetical* properties, which do not necessarily derive from its geometric origins. Try exploring some of these properties.

(6) In what ways can this exploration be continued? Perhaps with some objects other than straight lines? Or by venturing into three dimensions? Or by studying aspects other than just the number of regions? Find some ways on your own, and continue the study. Happy exploring!

We shall continue our exploration of sequences and related topics in future editions of this column.