# Spoof Numbers and Spoof Solutions - Part I

V G TIKEKAR

In this two-part article, we consider the curious notion of spoof numbers and spoof solutions which we get when we partially relax the conditions needed to define particular number families.

In number theory, we study various families of numbers: prime numbers, composite numbers, triangular numbers, square numbers, Mersenne numbers, Ramanujan numbers, and so on. Each family has its own characteristic properties. In exhibiting an instance of a particular type of number, we have to be careful that none of the required properties is violated. This observation leads to the following definition.

**Definition 1.** While constructing a number belonging to a particular family, if we relax some of the required properties or rules of formation but ensure that all the other properties of that family are satisfied, then such a number is called a *spoof number* of that family. Sometimes, such a number is also called a *quasi number* of that family.

We can similarly define the notion of *spoof solutions* by considering the spoof numbers obtained in the context of solutions of equations.

We shall describe two important situations which give rise to spoof numbers and spoof solutions, related to *perfect numbers* and *Fermat's Last Theorem*, respectively.

The notion of spoof numbers can be applied to any type of number, but is most often used for those types of numbers that are not known to exist, at least at present.

*Keywords: Spoof number, spoof solution, Fermat number, perfect number, Mersenne number, triangular number, Euler*

**Perfect numbers & non-perfect numbers**

**Definition 2** (Perfect number)**.** A positive integer is called a *perfect number* if all its divisors add up to twice that integer.

**Example 1.** 496 is a perfect number since all its divisors, namely, 1, 2, 4, 8, 16, 31, 62, 124, 248 and 496 add up to $992 = 2 \times 496$.

Readers can verify that 6, 28 and 8128 are other examples of perfect numbers. Euclid was aware of these four perfect numbers. (We remark here that the notion of perfectness of numbers goes back to the Greeks, who were fascinated by such properties.)

Leonard Euler, one of the greatest mathematicians the world has known (and certainly one of the most prolific) defined what he called the 'sigma ($S$) function.' It is very useful in handling perfect numbers.

**Definition 3** (Euler's sigma function)**.** If $n$ is a positive integer, then $S(n)$ = sum of all the divisors of $n$.

**Example 2.** $S(6) = 1 + 2 + 3 + 6 = 12$ and $S(8) = 1 + 2 + 4 + 8 = 15$.

So, using Definition 2, 6 is a perfect number, while 8 is not. Using Euler's $S$-function, the formal definition of a perfect number becomes:

**Definition 4** (Perfect number)**.** A positive integer $n$ is called a 'perfect number' if $S(n) = 2n$.

**Example 3.** $S(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28$. So 28 is a perfect number.

**Two important rules concerning the $S$ function.** Euler proved two important properties of the $S$ function that come of great use when working with this function. We state them below and illustrate them with numerical examples, but leave the proofs to the reader.

*Rule* 1. $S(a \times b) = S(a) \times S(b)$ iff $a$ and $b$ are coprime. (Note that the rule extends in an obvious way to the product of more than two coprime numbers.)

**Example 4.** Since 3 and 8 are coprime, $S(3 \times 8) = S(3) \times S(8)$, so

$$S(3 \times 8) = (1 + 3)(1 + 2 + 4 + 8) = 4 \times 15 = 60.$$

Direct verification: $S(3 \times 8) = S(24) = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60$.

*Rule* 2. If $p$ is any prime number and $a$ is any positive integer, then

$$S(p^a) = 1 + p + p^2 + p^3 + \ldots.. + p^a.$$

**Example 5.** As 2 is prime and 3 is a positive integer,

$$S(2^3) = 1 + 2 + 2^2 + 2^3 = 15.$$

Direct verification: $S(2^3) = S(8) = 1 + 2 + 4 + 8 = 15$.

We shall not give the proofs of these two rules here as they are available in all standard texts in number theory. The reader may refer to [3] for the proofs (the section 'Properties').

**Combining the two rules.** Now we present two examples in which we make use of both the rules given above.

**Example 6.** To compute $S(28)$:

$$\begin{aligned} S(28) &= S(4\times 7) = S(4)\times S(7) && \text{(by Rule 1, as 4 and 7 are coprime)}\\ &= S(2^2)\times S(7^1)\\ &= (1+2+2^2)\times(1+7^1) && \text{(by Rule 2)}\\ &= 7\times 8 = 56 = 2\times 28, \end{aligned}$$

implying that 28 is a perfect number.

**Example 7.** To compute $S(40)$:

$$\begin{aligned} S(40) &= S(5\times 8) = S(5)\times S(8) && \text{(by Rule 1, as 5 and 8 are coprime)}\\ &= S(5^1)\times S(2^3)\\ &= (1+5)\times(1+2+2^2+2^3) && \text{(by Rule 2)}\\ &= 6\times 15 = 90 \neq 2\times 40, \end{aligned}$$

implying that 40 is not a perfect number.

**Remark.** All the perfect numbers that we have met till now are even. Further, we note that:

(i) Every known perfect number is of the form $2^{p-1}\times(2^p-1)$, where both $p$ and $2^p-1$ are prime numbers;

(ii) The smallest known perfect number (namely, 6) corresponds to the case $p=2$ in the above formula, and the largest currently known perfect number corresponds to the case $p=82589933$.

**Spoof perfect numbers**

We shall now relax some of the rules and/or assume some incorrect properties to generate spoof perfect numbers.

**Example 8.** Let us assume (incorrectly, of course) that 4 is prime. Then we can get $60 = 3\times 4\times 5$ as the prime factorisation of 60. This would mean that

$$\begin{aligned} S(60) &= S(3)\times S(4)\times S(5) && \text{(by applying Rule 1, as 3, 4, 5 are coprime)}\\ &= (1+3)(1+4)(1+5) && \text{(by applying Rule 2, since 4 is taken to be prime)}\\ &= 120 = 2\times 60. \end{aligned}$$

Thus 60 satisfies the property of perfectness **if** we assume 4 to be a prime number.

But 4 is not really prime as we know. Further, the sum of all the divisors of 60 is

$$S(60) = 1+2+3+4+5+6+10+12+15+20+30+60 = 268 \neq 2\times 60.$$

So we cannot correctly call 60 a perfect number.

To accommodate such a situation, we create a new terminology and call 60 a 'quasi' or a 'spoof' perfect number under the assumption that 4 is prime. This indicates that 60 imitates the behaviour of a genuine perfect number under certain circumstances, namely, by treating 4 to be prime. Further, since 60 is even, we call it an "even spoof perfect number under the condition that 4 is prime."

**Definition 5** (Spoof prime, quasi prime)**.** A number that we incorrectly assume to be prime in some situation is called a *spoof prime* or a *quasi prime* in the context of that situation.

For example, in Example 8 above, 4 is a quasi or a spoof prime (in the context of Example 8). In the following, we shall meet many such quasi primes (which are so designated only in the context of the corresponding situations).

**Example 9.** Here we shall again check 60 for perfectness, just as we did in Example 8, but instead of 4, we shall assume (incorrectly) that 10 is prime. Proceeding as earlier, we get:

$$\begin{aligned} S(60) &= S(2) \times S(3) \times S(10) \\ &= (1+2)(1+3)(1+10) = 132 \neq 2 \times 60. \end{aligned}$$

So 60 is *not* a spoof perfect number if we assume that 10 is prime.

From Examples 8 and 9, it should be absolutely clear that merely saying "The number $n$ is a spoof perfect number" is not completely correct. We must explicitly state the rule that has been relaxed. Without this detail, calling a particular number a spoof number is an incomplete statement and hence not correct. To emphasize this point, let us state again: 60 *is a spoof perfect number if we assume that* 4 *is prime, but* 60 *is not a spoof perfect number if we assume that* 10 *is prime.*

We shall study more examples of this new category of numbers.

**Example 10.** We examine the number 90, assuming (again incorrectly) that 9 is prime. We have:

$$\begin{aligned} S(90) &= S(2 \times 5 \times 9) = S(2) \times S(5) \times S(9) && \text{(by Rule 1)} \\ &= (1+2) \times (1+5) \times (1+9) && \text{(by Rule 2)} \\ &= 180 = 2 \times 90. \end{aligned}$$

Hence 90 is a spoof perfect number if we assume 9 to be prime.

But if we assume (incorrectly) that 10 is prime (rather than 9), then:

$$\begin{aligned} S(90) &= S(3^2) \times S(10) && \text{(by Rule 1)} \\ &= (1+3+3^2) \times (1+10) && \text{(by Rule 2)} \\ &= 143 \neq 2 \times 90. \end{aligned}$$

Hence 90 is not a spoof perfect number if we assume that 10 is prime.

**Example 11.** We examine the number 84. We have:

$$\begin{aligned} S(84) &= S(2^2) \times S(3) \times S(7) \\ &= (1+2+2^2) \times (1+3) \times (1+7) = 224 \neq (2 \times 84) \end{aligned}$$

Hence 84 is not a perfect number.

But if we assume 6 to be a prime number, then we get

$$\begin{aligned} S(84) &= S(2) \times S(6) \times S(7) \\ &= (1+2)(1+6)(1+7) = 168 = 2 \times 84, \end{aligned}$$

so 84 is a spoof perfect number if 6 is considered to be prime.

**Example 12.** We examine the number 12. We have:

$$\begin{aligned} S(12) &= S(2^2) \times S(3) \\ &= (1 + 2 + 2^2) \times (1 + 3) = 28 \neq (2 \times 12). \end{aligned}$$

If we assume 4 to be a prime number, then:

$$\begin{aligned} S(12) &= S(3) \times S(4) \\ &= (1 + 3) \times (1 + 4) = 20 \neq (2 \times 12). \end{aligned}$$

We see that the number 12 is neither a perfect number nor a spoof perfect number under the assumption that 4 is prime.

**Example 13.** We know (from Example 6) that 28 is a perfect number. Now, if we assume that 4 is prime, then:

$$\begin{aligned} S(28) &= S(4) \times S(7) && \text{(by Rule 1)} \\ &= (1 + 4) \times (1 + 7) && \text{(by Rule 2)} \\ &= 40 \neq (2 \times 28) \end{aligned}$$

Hence 28 is a perfect number, but it is *not* a spoof perfect number if we take 4 to be a prime number.

**Example 14.** We examine the number 840. We are going to assume (incorrectly) that both 4 and 6 are prime numbers.

$$\begin{aligned} S(840) &= S(4) \times S(5) \times S(6) \times S(7) && \text{(by Rule 1)} \\ &= (1 + 4) \times (1 + 5) \times (1 + 6) \times (1 + 7) && \text{(by Rule 2)} \\ &= 1680 = 2 \times 840 \end{aligned}$$

Hence 840 is a spoof perfect number under the assumption that 4 and 6 are prime numbers.

**Example 15.** We leave it to the reader to verify that the integer

$$390405312000 = 4 \times 8 \times 9 \times 10 \times 15 \times 22 \times 46 \times 94 \times 95$$

is an even spoof perfect number if one wrongly assumes that all the factors stated on the right side are primes.

**Remark.** We close the article by noting that all the spoof perfect numbers that we have seen till now are even numbers, i.e., they are even spoof perfect numbers.

What about odd perfect numbers and odd spoof perfect numbers? We examine this question in Part II of this article.

## References


1. Jason Chuang (Quanta Magazine), "Mathematicians Open a New Front on an Ancient Number Problem", https://www.quantamagazine.org/mathematicians-open-a-new-front-on-an-ancient-number-problem-20200910/
2. Wikipedia, "Descartes number" from https://en.wikipedia.org/wiki/Descartes_number
3. Wikipedia, "Divisor function" from https://en.wikipedia.org/wiki/Divisor_function



**PROF. V.G. TIKEKAR** retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.