ClassRoom

# Areas of Triangles between Two Parallel Lines with Same Base are Equal

## Proof(?) by Paper Folding

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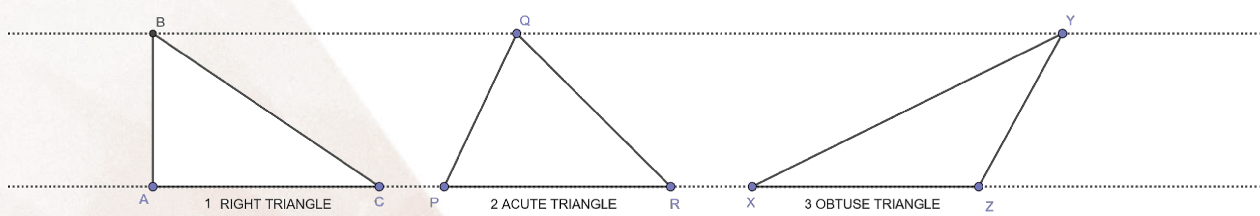
### Introduction

There are many methods to prove that the areas of triangles between two parallel lines and with the same base length are equal. Here, we are discussing the proof using a paper folding method. We give the geometric reasoning of the proof and follow it up with the implications of this in the paper folding activity. During the paper folding activity, some geometrical properties are used intuitively, and the questions alongside will serve to stimulate both observation as well as reasoning and deductive skills in the students. We hope that this approach illustrates that results which are observed during the paper folding are based on beautiful and rigorous mathematics.



In this article, we consider three triangles (right-angled, acute and obtuse), all of which are of the same height and the same base. We prove that all these triangles have the same area. Note that the following are **not** assumed:

1. The formula for the area of a triangle. This will be arrived at (for all three types of triangles).
2. The mid-point theorem.

### Theorem to be proved

Triangles on the same base (or equal bases) and between the same parallel lines (i.e., of equal height) are equal in area.

### Steps

Here we are considering three different types of triangles, i.e., the right triangle, acute triangle and obtuse triangle with the same base length and lying between the same parallel lines. For this we construct each of these triangles with base '$b$' and height '$h$' as shown in Figure 1.

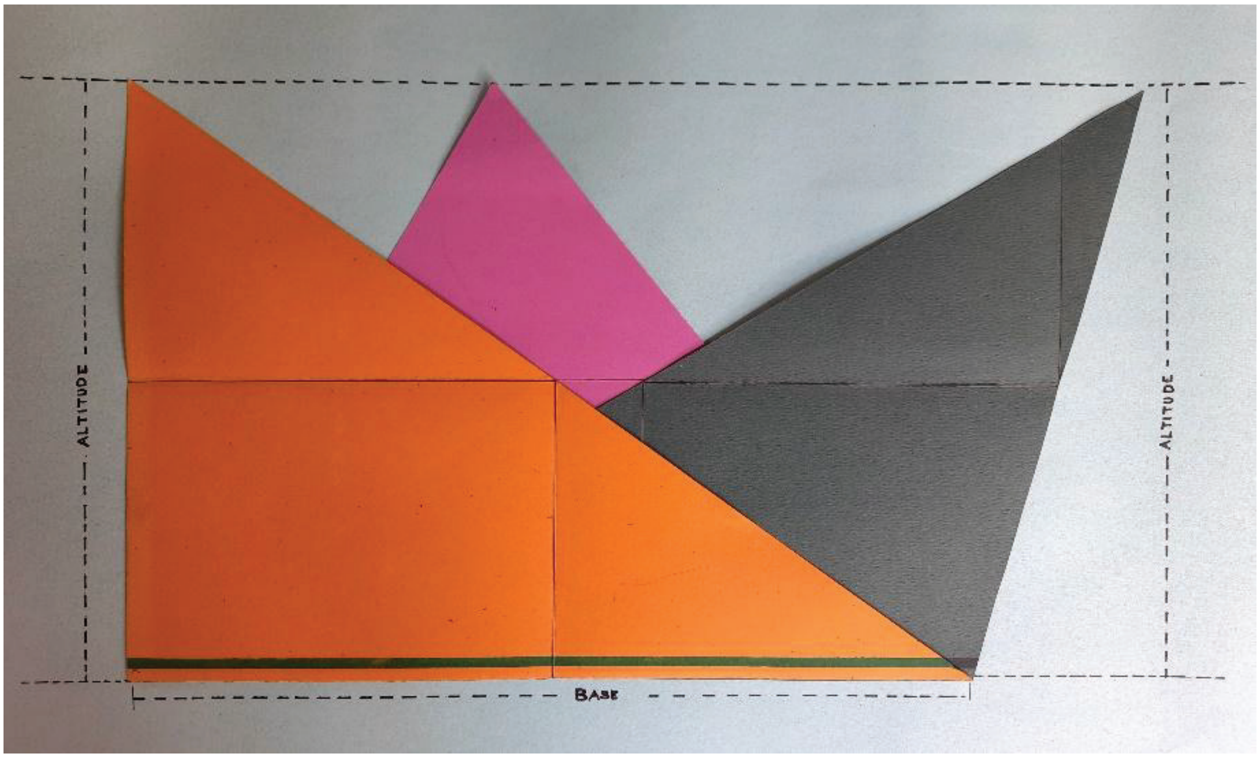



Figure 1.

Let us follow the steps for each triangle as given below.

(Note: The sequence of reasoning is to be read column-wise.)

## 1. Right Triangle

**Geometric Reasoning**

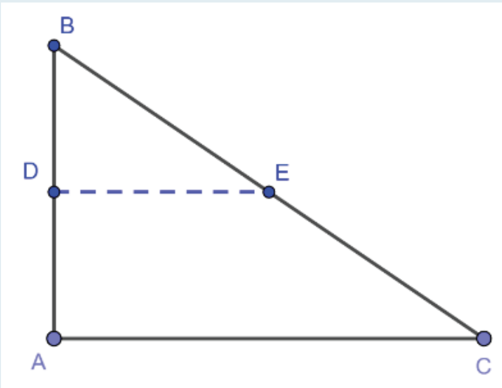



Figure 1.

Let $\Delta ABC$ be a triangle (of base $AC = b$ and height $AB = h$) which is right-angled at $A$. Let $DE$ be the perpendicular bisector of $BA$. We will prove that $E$ is the mid-point of $BC$.

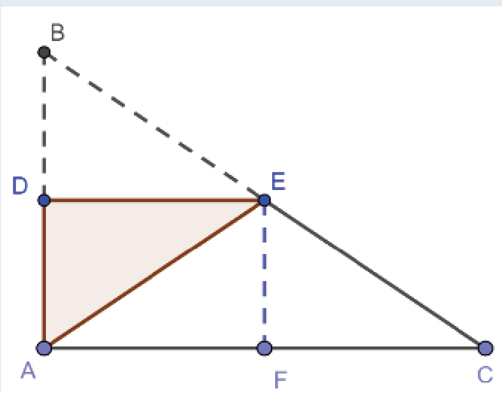



Figure 2.

Drop $EF$ perpendicular to $AC$ and join $AE$.

Clearly, $AFED$ is a rectangle since three of its angles are right angles.

$$\therefore DB = DA = EF.$$

And $DE$ is parallel to $AF$ (and hence to $AC$).

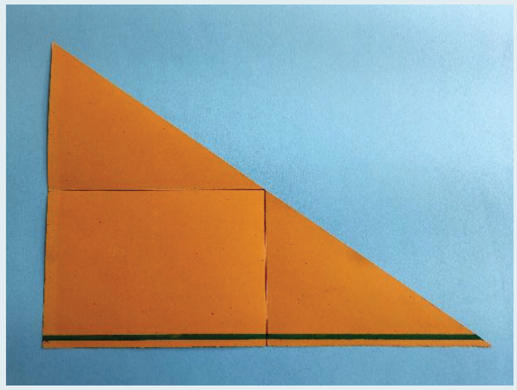
So, $\measuredangle DEB = \measuredangle FCE$ and this proves that $\Delta DEB \cong \Delta FCE$ (by $AAS$ since both are right triangles).

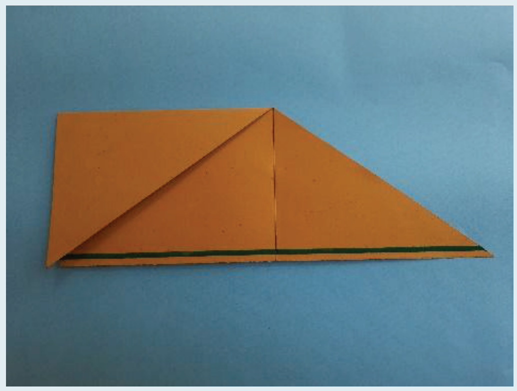
$$\therefore BE = EC$$

So $E$ is the mid-point of $BC$.

**Paper Folding**

Make a paper cut-out of the right triangle $\Delta ABC$.

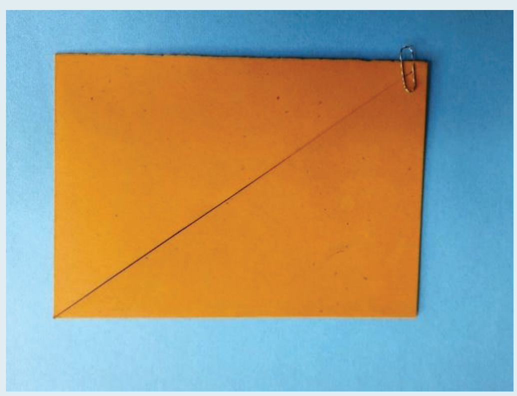




Fold the triangle so that the point $B$ coincides with $A$. The fold line $DE$ is the perpendicular bisector of $BA$.

Fold $AC$ so that (i) the two parts of $AC$ are aligned and (ii) the fold line passes through $E$

Let the point where the fold line meets $AC$ be $F$.



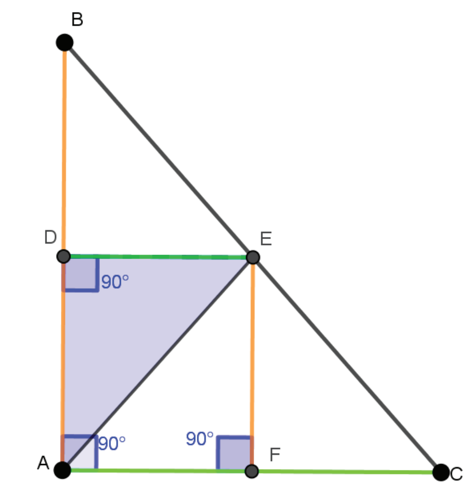
<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
<tr><td>
We have just reasoned geometrically that $E$ is the mid-point of $BC$. Can you also prove that the four triangles $ADE$, $BDE$, $EFA$ and $EFC$ are congruent?<br>
$$BD = DA = EF = h/2$$
$$AF = FC = b/2$$
And, $AE = BE = CE$<br>


Figure 3
</td><td>
$\measuredangle AFE = \measuredangle CFE$ (since they superimpose on each other)<br>
$$= \frac{1}{2}\measuredangle AFC = \frac{1}{2} \times 180^\circ$$
$$= 90^\circ \therefore EF \perp AC$$
Observe that $C$ coincides with $A$ superimposing $\Delta EFC$ on to $\Delta EFA$, while $\Delta BDE$ superimposes on $\Delta ADE$ thanks to the fold along $DE$. So, we get two rectangles with base $b/2$ and height $h/2$. The sum of the areas of these two rectangles is equal to the area of $\Delta ABC$.[1]
</td></tr>
<tr><td colspan="2">
The equalities we arrived at by geometric reasoning resonate in the paper folding. We see that the area of the rectangle<br>
$$DEFA = AF \times AD,$$
$$AF = \frac{1}{2}AC = b/2 \text{ and } AD = \frac{1}{2}AB = h/2,$$
$$\Delta ABC = 2 \times DEFA = 2 \times AF \times AD = 2 \times b/2 \times h/2 = \frac{1}{2}b \times h$$
</td></tr>
<tr><td colspan="2">
We also observe both through geometry and through paper folding that $BE = AE = CE \Rightarrow E$ is the circumcentre $\Rightarrow$ We have also arrived at the result that the circumcentre of a right triangle is the midpoint of its hypotenuse!
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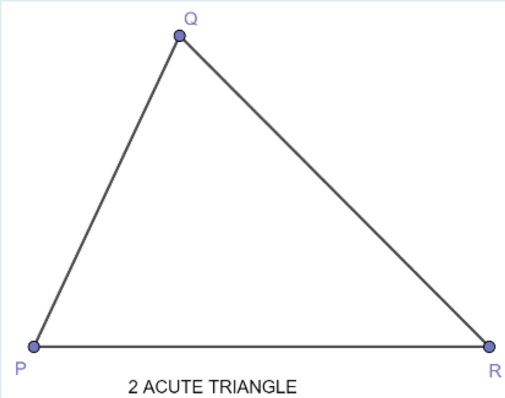
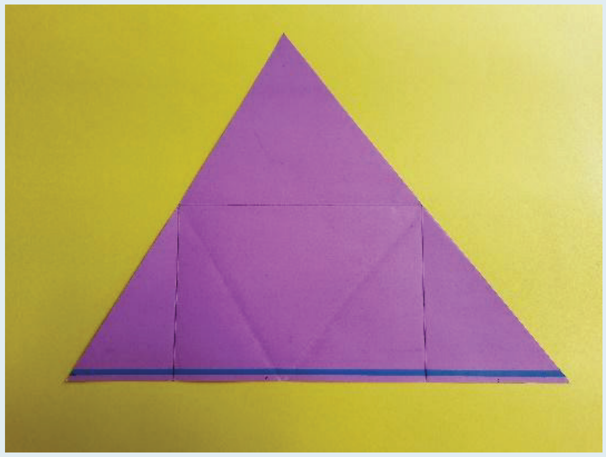
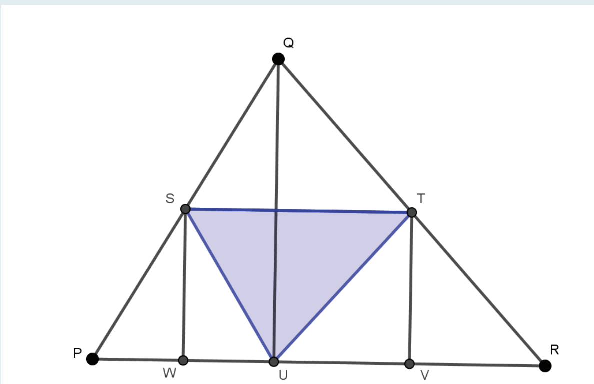
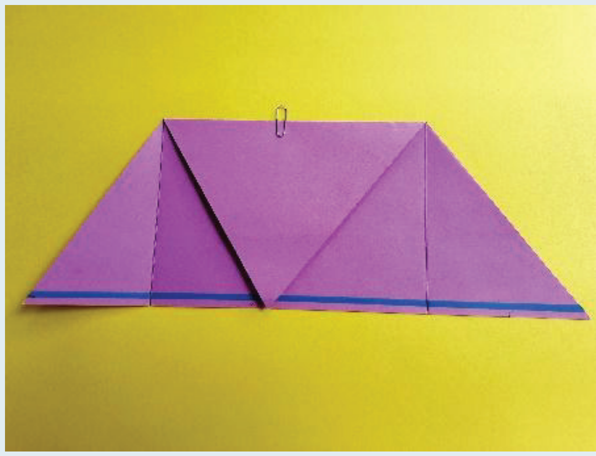
</table>

Now we can check the other triangles with similar steps.

1 Check Unfolding, At Right Angles, Jul 2014 issue for folding a perpendicular to a given line through a given point.

## 2. Acute[2] Triangle

| Geometric Reasoning | Paper Folding |
|---|---|
|   |  |
| Let $\Delta PQR$ be an acute angled triangle with base $PR = b$ and height $h$. | |
|   Figure 4<br><br>Here, we mark the mid-points $S$ and $T$ of sides $QP$ and $QR$ respectively and draw the line $ST$. Draw two perpendiculars $SW$ and $TV$ to the base $PR$. Let $U$ be the foot of the perpendicular drawn from $Q$ to $PR$. ($QU = h$)<br><br>In the right triangle $\Delta QUP$, $S$ is the mid-point of the hypotenuse $QP$. | <br><br>Mark the mid-points $S$ and $T$ of the sides $PQ$ and $QR$ by folding.<br><br>Mark the foot $U$ of the perpendicular to $PR$ through $Q$ by folding.<br><br>Can you see why $Q$ coincides with $U$ on folding along $ST$? |

2 This also works for right and obtuse triangles and for those cases we need to consider the longest side as the base. However, for an acute triangle the base can be any side.

<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
<tr><td>
We have proved above that therefore $SQ = SP = SU$.<br><br>
Similarly, in the right triangle $\Delta QUR$, $T$ is the mid-point of the hypotenuse $QR$.<br><br>
So, $TQ = TR = TU$.<br><br>
$SQ = SU$, $TQ = TU \therefore SQTU$ is a kite. Its diagonals $QU$ and $ST$ will be perpendicular. If they intersect at $Y$, then $QY = YU = h/2$.<br><br>
$ST$ is the perpendicular bisector of $QU$.<br><br>
We can prove that $\Delta SWP$ and $\Delta SWU$ are congruent. (How?)<br><br>
So, $W$ is the mid-point of $PU$.<br><br>
Similarly, $\Delta TVU \cong \Delta TVR$.<br><br>
So, $V$ is the mid-point of $UR$.<br><br>
$\therefore ST = WV = WU + UV = \frac{1}{2}PU + \frac{1}{2}UR$<br>
$= \frac{1}{2}(PU + UR) = \frac{1}{2}PR = b/2$
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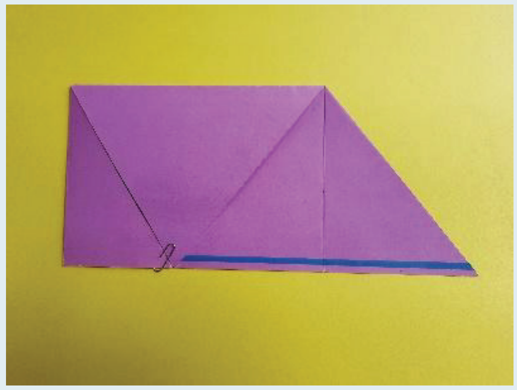<br>
<br><br>
Can you also see why $P$ and $R$ fold along $SW$ and $TV$, the perpendiculars to $PR$, to meet at $U$?<br><br>
[What kind of triangles are $\Delta PSU$ and $\Delta RTU$?]
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Again, the paper folding aligns perfectly with the geometric reasoning.<br>
$$ST \perp QU \text{ and } QU \perp PR \Rightarrow ST || PR$$
$ST||PR$, and $SW, TV \perp PR \Rightarrow WSTV$ is a rectangle, and its area is $b/2 \times h/2$<br><br>
Since $WSTV$ is $\Delta PQR$ folded into 2 layers, $\Delta PQR = 2 \times STVW = \frac{1}{2}bh$.
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<tr><td colspan="2">
Find three angles which are equal to $\measuredangle PQR$, $\measuredangle QRP$ and $\measuredangle RPQ$ respectively. Using these three angles can you prove that the sum of the angles in a triangle is $180°$ ?<br>
$$\measuredangle PQR = \measuredangle SUT \text{ since } USQT \text{ is a kite}$$
$$\measuredangle QRP = \measuredangle TUV \text{ since } \Delta TRV \cong \Delta TUV$$
$$\measuredangle RPQ = \measuredangle SUW \text{ since } \Delta PSW \cong \Delta USW$$
$\therefore \measuredangle PQR + \measuredangle QRP + \measuredangle RPQ = \measuredangle SUT + \measuredangle TUV + \measuredangle SUW = \measuredangle WUV = 180°$<br><br>
We have arrived at the result that the sum of the angles of this acute angled triangle is $180°$ !<br><br>
Note that the sum of the angles for any triangle can be explored this way provided that the longest side is considered the base.
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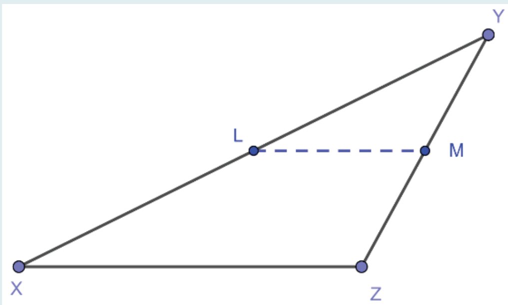
## 3. Obtuse Triangle

**Geometric Reasoning**

Let $\Delta XYZ$ be an obtuse angled triangle with base $XZ = b$ and height $h$.




Mark the mid-points $L$ and $M$ of the sides $XY$ and $YZ$ respectively of the obtuse angled triangle $\Delta XYZ$.

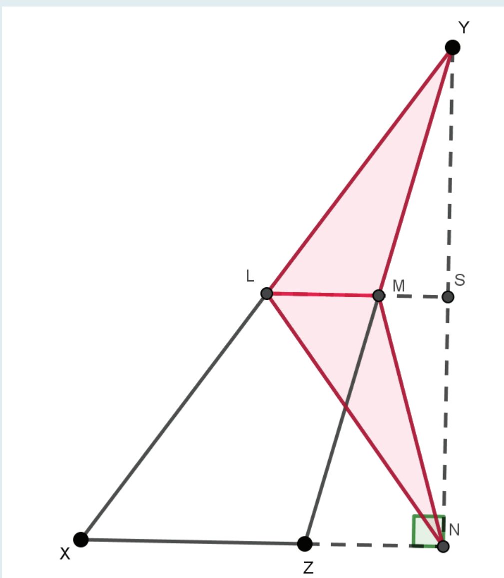
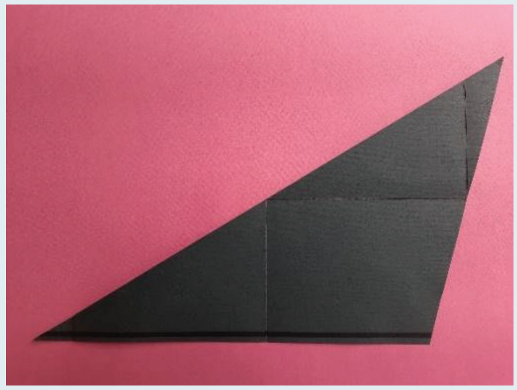



Figure 5

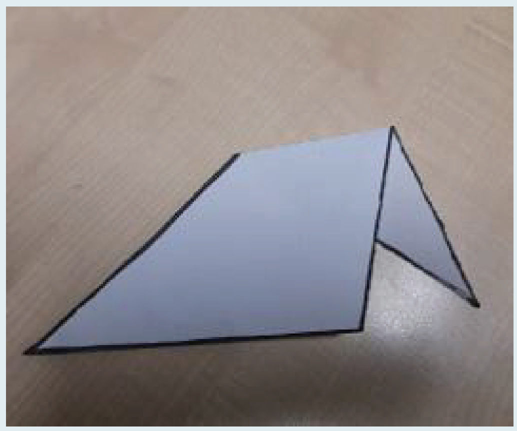
Let $N$ be the foot of the perpendicular from $Y$ to $XZ$ extended i.e. $YN \perp XN$. Let $LM$ extended meet $YN$ at $S$.

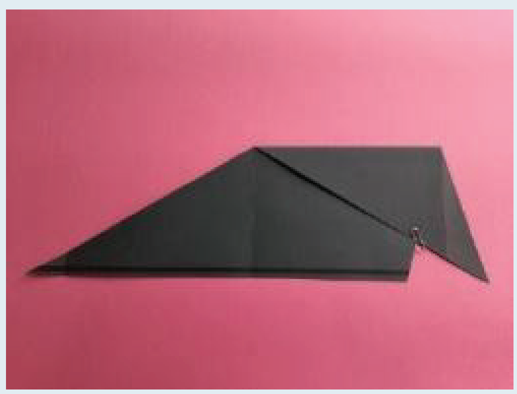
We can prove that $LS$ is the perpendicular bisector of $YN$ in the following steps:

**Paper Folding**



By folding, mark the mid-points $L$ and $M$ of the sides $XY$ and $YZ$ of the triangle $XYZ$. Fold[3] along $LM$. It helps to fold it back rather than front.





3 Folding $\Delta LMY$ back may help with the subsequent folds (see blue triangle in Figure 8).

## Geometric Reasoning

1. $\Delta XYN$ is a right triangle and $L$ is the midpoint of its hypotenuse $XY$,

$$\therefore LX = LY = LN$$

$\Delta ZYN$ is a right triangle and $M$ is the midpoint of its hypotenuse $ZY$,

$$\therefore MZ = MY = MN$$

2. $\Delta LMY \cong \Delta LNM$ by $SSS$

$$\Rightarrow \measuredangle MLY = \measuredangle MLN$$

3. $\therefore \Delta LSY \cong \Delta LSN$ by $SAS$

$\Rightarrow YS = SN$ and $\measuredangle LSY = \measuredangle LSN = \frac{1}{2} \times 180^\circ = 90^\circ$ i.e. $LS$ is the perpendicular bisector of $YN$

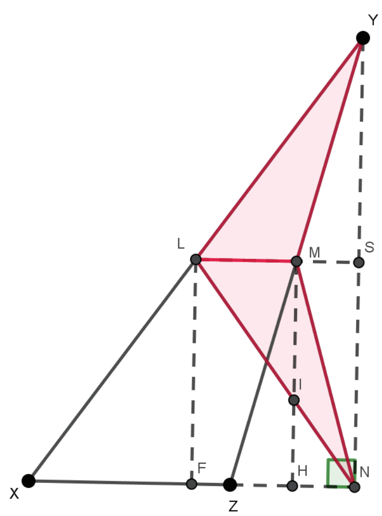


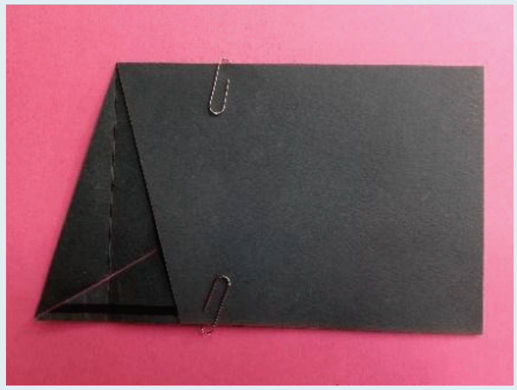

Figure 6

Drop perpendicular $LF$ to side $XZ$.

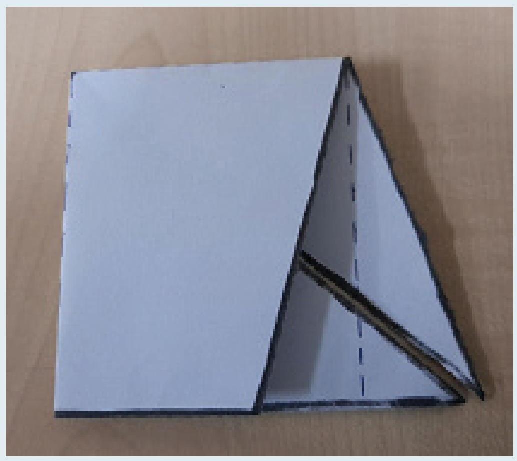
$\Delta LXN$ is isosceles, $\therefore LF$ is the angle bisector of $\measuredangle XLN$, in fact it is the line of symmetry of $\Delta LXN$, $\therefore LF$ is perpendicular bisector of $XN$

## Paper Folding

Our geometric reasoning proves that $Y$ coincides with $N$ when the triangle is folded along $LM$.



So, when you fold along $LF \perp XZ$, where does $X$ end up? Note that the folded triangle becomes a trapezium with 2 adjacent right angles. Depending on the first fold along $LM$, these right angles may be on the right or on the left as we have shown above (and below).



Now if you fold along $MH$, can you see why the folded triangles $\Delta NIM$ and $\Delta NIH$ fit the remaining space beside the trapezium $LMZF$?

<table>
<tr><th>Geometric Reasoning</th><th>Paper Folding</th></tr>
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Drop perpendicular $MH$ to side $XZ$ (produced). Let $LN$ intersect $MH$ at $I$.

Since $\Delta ZMN$ is isosceles, $MH$ is the line of symmetry of $\Delta ZMN$

$\Rightarrow \Delta ZHM \cong \Delta NHM$ by $RHS \Rightarrow ZH = HN$

$\Rightarrow \Delta ZIM \cong \Delta NIM$ and $\Delta ZIH \cong \Delta NIH$ (Why?)

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Again, the paper folding aligns perfectly with the geometric reasoning.

Now, LM $||FH$, which is part of $XZ$ (extended), $LF||MH$ since both are perpendicular to $XZ$ (produced), $\therefore$ $LMHF$ is a parallelogram.

Since $LF \perp XZ$, $LMHF$ is in fact a rectangle with area $b/2 \times h/2$ because,

its base $FH = FN - HN = \frac{1}{2}XN - \frac{1}{2}ZN = \frac{1}{2}(XN - ZN) = \frac{1}{2}XZ = b/2$ and

its height $LF = SN = \frac{1}{2}YN = h/2$

Since $LMHF$ is $\Delta XYZ$ folded into 2 layers, $\Delta XYZ = 2 \times LMHF = \frac{1}{2}bh$.

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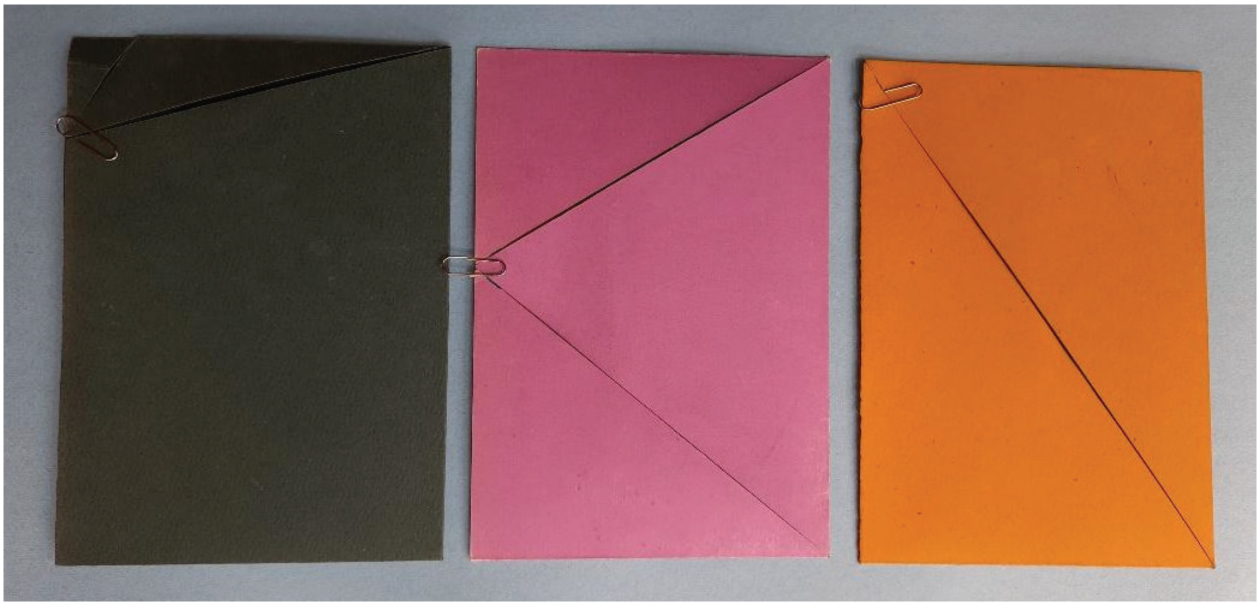
</table>

A critical question at this point is whether $F$, the foot of the perpendicular from $L$ to $XZ$, would always be within $XZ$. We strongly encourage our readers to explore the position of $F$ using GeoGebra (or otherwise) by varying the position of $Y$ without changing the height of the triangle.

If $F$ is outside $XZ$ i.e. outside the paper triangle, can these steps be modified to get a double-layered rectangle? If so, how? We hope to discuss these in a future article.



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