ClassRoom

# My Square is your Triangle! – The King said to Pascal

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## I. Introduction

Imagine a lonely King placed in the left bottom corner of a $n \times n$ chess board! We are aware that the King can move only one square in any possible direction on the chess board. The King should reach the right top corner of the chess board by choosing his own path. See Figure 1.

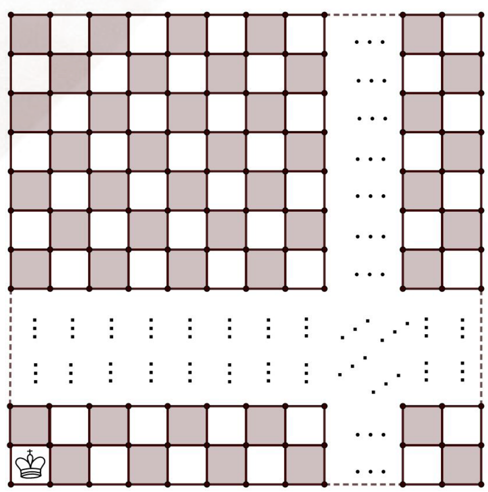

Figure 1

Let us define a path to be a combination of possible moves between the initial position of the King (left bottom corner) and the final position of the King (right top corner). Here's

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an interesting question! *How many paths are there for the King to reach his destined position?* **Answer:** Infinite! Why? Because the King can keep moving back and forth and that leads to infinite choices!

We can keep the interest of the question alive by imposing certain conditions in the movement of the King as follows:

- The King is allowed to move either to the right or up only.
- No other movement such as left, down, diagonal is allowed.

## II. Planner

First let us make a study with smaller values of $n$ such as 2, 3 and 4. We can make observations and notes side by side while studying the paths of the King on $2 \times 2$, $3 \times 3$ and $4 \times 4$ chess boards. Later, we can establish the general theory.

**$2 \times 2$ Chess Board.** The King is placed in the left bottom corner of a $2 \times 2$ chess board as shown.

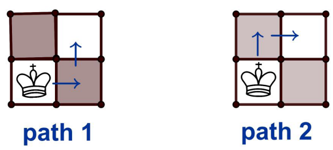


Figure 2

There are exactly two paths for the King to reach the right top corner. Note that each path is a two movement path for this to happen. See Figure 2.

**Key observation.** In a $n \times n$ chess board, the King can move from any square to his immediate diagonal (right and up) square in exactly two ways.

**$3 \times 3$ Chess Board.** The King is placed in the left bottom corner of a $3 \times 3$ chess board as shown.

There are exactly six paths for the King to reach the right top corner. Note that each path is a four movement path for this to happen. See Figure 3.

Remember the conditions: the King can move either right or up only and no other move such as left, down or diagonal is allowed. Observe that the King enters the destined right top corner finally through the square just below it in paths 1;4; 6.

Also observe that the King enters his destination through the square to its immediate left in paths 2;3; 5.

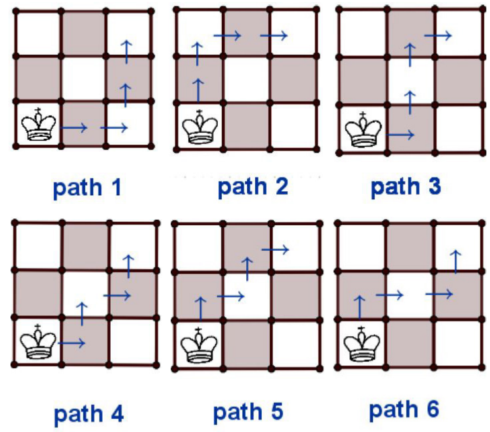


Figure 3

### Key observations

- The King can enter the destined corner square either from the square to its immediate left or the square immediately below it; there are no other possibilities.
- The King can enter any intermediate square either from the square to its immediate left or the square immediately below it; there are no other possibilities.
- The King can enter any boundary square in the first column by an upward movement only (as there is no column to its left).
- The King can enter any boundary square in the bottom row by a movement to the right only (as there is no row below it).

**$4 \times 4$ Chess Board.** The King is placed in the left bottom corner of a $4 \times 4$ chess board, as shown.

There are ten paths for the King to reach the right top corner through the square just below it (Figure 4), and there are ten paths for the King to reach the right top corner through the square to its immediate left (Figure 5). There are thus exactly twenty paths for the King to reach the right top corner. Note: Each path is a six movement path for this to happen.

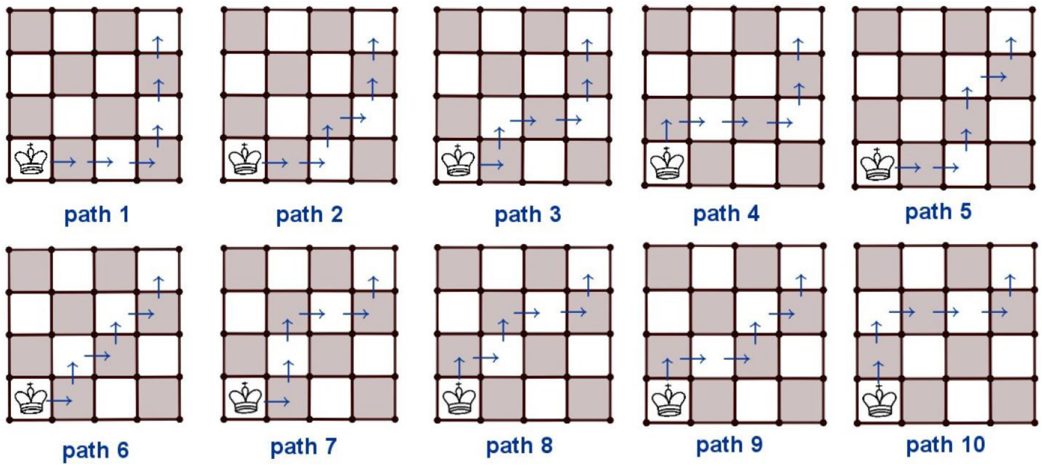


Figure 4

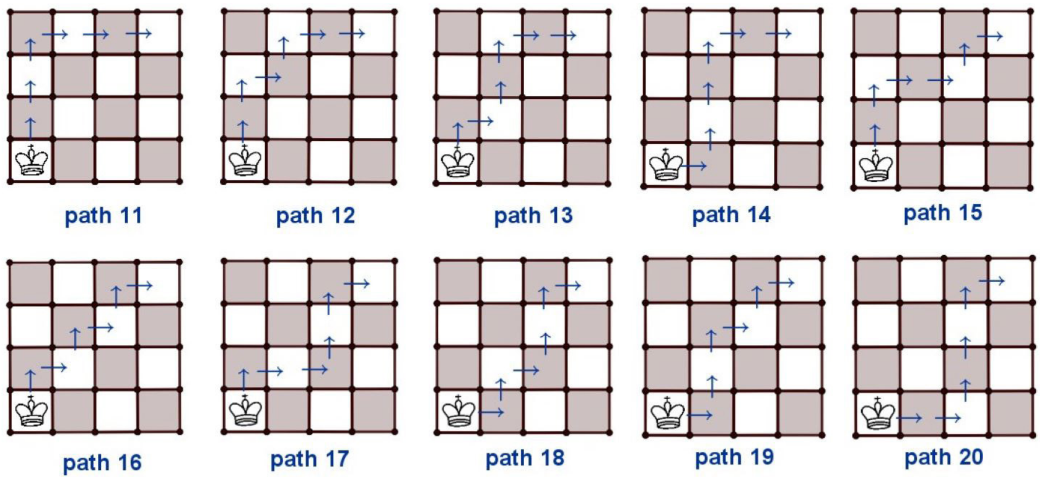


Figure 5

## Key notes

- We observe that the first ten paths and the next ten paths are correspondingly symmetric. The up and right movements in the first ten paths are respectively replaced by right and up movements in the next ten paths.
- We also observe that the King's final move is either an up movement in exactly ten paths or a right movement in exactly ten paths.

## III. Theory: Assigning Weights

Now, we have reached a stage for setting the theory. Let us not indulge in drawing paths any more! We assign weights to every square in the chess board other than the initial left bottom corner square, the starting position of the King.

## Assigning weights

- Each square is assigned the number that equals the total number of paths by which the King can reach that square.
- Every square along the boundary of the first column or last row is assigned 1 as there is only one possible path for the King to reach the respective square. (See Figures 6 and 7.)

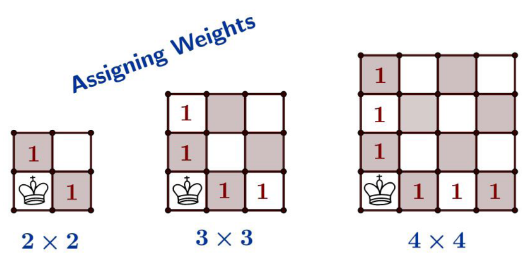


Figure 6

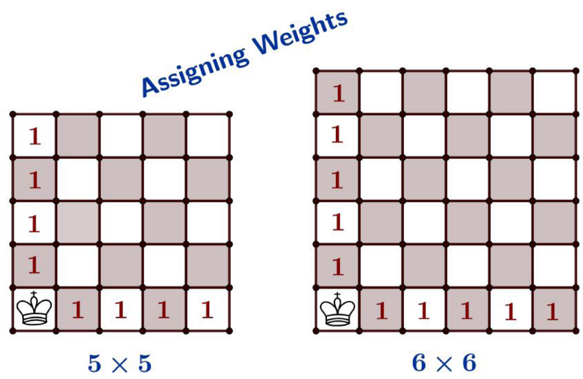


Figure 7

Remember that a path may have one or more moves! The King can reach any square in the first column or last row by only one path (way).

What about assigning weights to other squares? Let us try to develop a relation between assigned weights.

### Connecting weights

- Let $A$, $C$, $B$ be three squares in a $N \times N$ chess board assigned with weights $a$, $c$, $b$ respectively, as shown (Figure 8).
- For the King to reach square $C$, its previous position is either square $A$ or square $B$ only.
- There are $a$ ways (paths) for the King to reach square $A$ from his left bottom corner; and only one way (right move) from square $A$ to square $C$.
- Therefore there are $a \times 1 = a$ ways (paths) for the King to reach square $C$ from his left bottom corner through square A.
- There are $b$ ways (paths) for the King to reach square $B$ from his left bottom corner; and only one way (up move) from square $B$ to square $C$.
- Therefore there are $b \times 1 = b$ ways (paths) for the King to reach square C from his left bottom corner through square $B$.
- Therefore $c = a + b$.

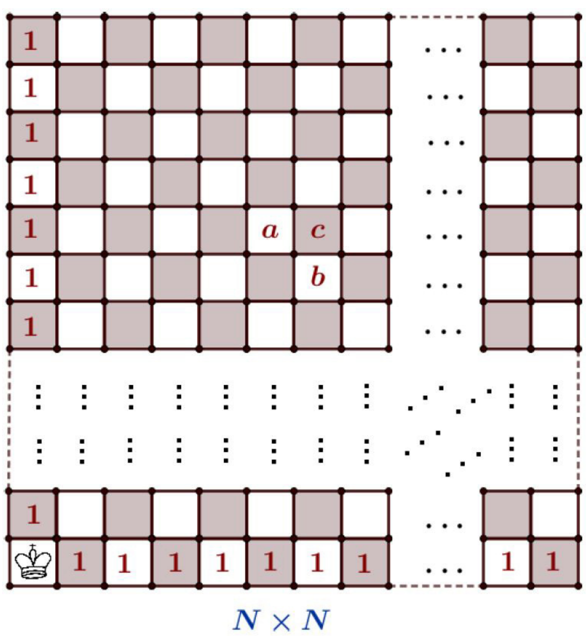


Figure 8

It follows that the weight of a square (not in first column or last row) equals the sum of the weights of its immediate left square and immediate below square.

**Inductively assigning weights.** We have the weights (1) for each square in the first column and last row. Having laid the foundation theory, we can construct the entire $N \times N$ chess board with weights inductively, for any $N$. This is like constructing a $N$-storey building with a strong foundation.

We can now start constructing weights for $2 \times 2$, $3 \times 3$, $4 \times 4$, $5 \times 5$, $6 \times 6$ chessboards and observe how the entire construction looks! See Figures 9 and 10.

Likewise, we can construct weights for any $N \times N$ chess board! Do you observe something interesting here ? YES! (See Figures 11 to 14.)

Now, we know why the King says to Pascal, "My Square is your Triangle!"

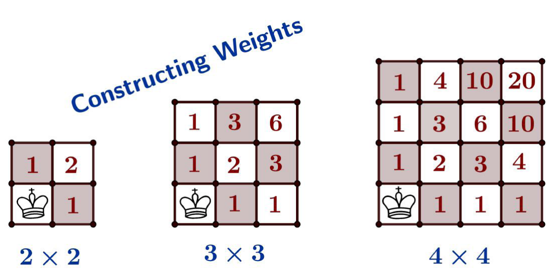


Figure 9

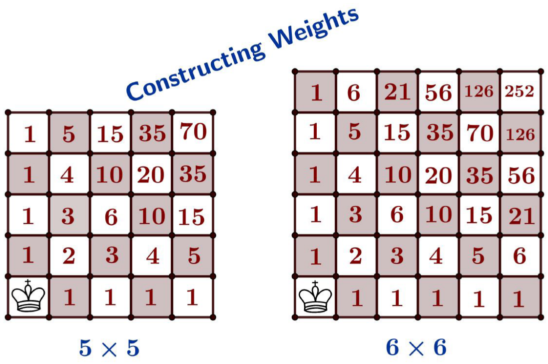


Figure 10

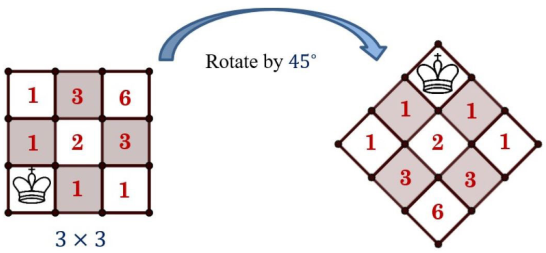


Figure 11

**Exercises**

- What is the total number of moves (not paths) in a 7 × 7 chess board for the King from left bottom corner to reach the right top corner?
- Among them, how many are up moves? how many are right moves?
- What is the weight of right top corner in a 7 × 7 chess board?
- Suppose the King makes *r, r, u, u, u, u* in a 6 movement path to reach an intermediate square in a 7 × 7 chess board. (Here, *r* means right move; *u* means up move.) Will the King reach a different square if the order of these 6 moves is changed?

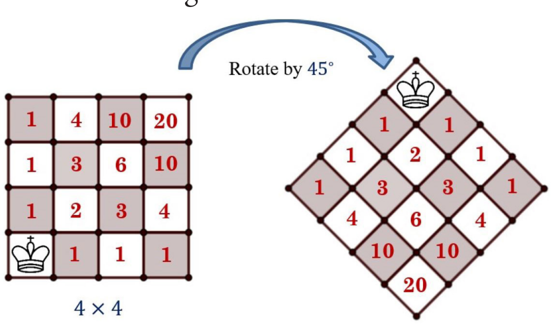


Figure 12

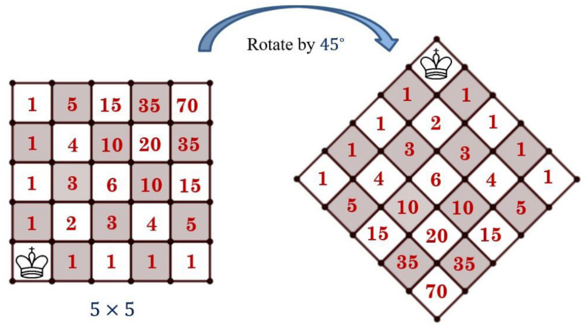


Figure 13

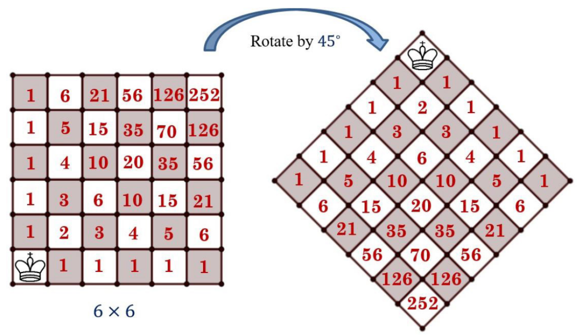


Figure 14

- Can we now say that the weight of a square attained by a 6 movement path with 2 right moves and 4 up moves is ${}^6C_2$ or ${}^6C_4$?
- What is the weight of a $n$ movement path having $r$ right moves and $n - r$ up moves in a $N \times N$ chess board?
- Can we make out combinatorial identities such as ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$, etc... from this setup?

**Projects for further exploration**

- Fix the number of moves between initial and final positions with no restrictions in the moves of the King. Find the total number of paths in such case.
- Make "No Entry" for certain squares between initial and final positions with no restrictions in the moves of the King and no repetition of positions (squares). Find the total number of paths in such case.

## References

[1] Wikipedia, "King (chess)" from https://en.wikipedia.org/wiki/King_(chess)

[2] https://www.geeksforgeeks.org/

[3] AMTI/NMTC/2019/Stage-1/Junior/Problem 23

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