Bachet's Problem

A.RAMACHANDRAN

There is a question, often posed as a puzzle or brainteaser, which has been popular for generations. It was probably in circulation earlier but came to be more widely known due to the Frenchman Claude Gaspar Bachet de Meziriac (1581-1638). The problem is not too easy to solve but the solution can be appreciated by anyone. It runs as follows:

A trader had a 40-pound standard weight which was dropped down accidentally and broke into four pieces, each weighing a different integral number of pounds. The trader then found that with the four weights thus obtained he could measure any integral number of pounds from 1 to 40, placing the weights on one, or both, of the pans of his balance. What are the weights of the four pieces?

Though I had come across the problem and its solution quite some time ago, I recently wondered how one could solve it by logical reasoning. I started by assuming the four weights to be a, b, c and d, with $a > b > c > d$.

Now, $b + c + d = 40 - a$. Let us assume that with the three weights b, c and d, one can measure all integer pound weights from 1 to $(b + c + d)$, both inclusive. Now, by taking weight a and some or all of b, c, d we can measure all integer weights from $(a + 1)$ to 40. Again, we can work downwards from a to obtain all integral weights till $a - (b + c + d) = a - (40 - a) = (2a - 40)$. But integers weights from $(b + c + d + 1)$ till $(2a - 41)$, both inclusive, cannot be obtained.

A numerical example to drive home this point: Let $a = 34$. Then $b + c + d = 6$. Let $b = 3$, $c = 2$ and $d = 1$. With these we can measure all the integer weights from 1 to 6. Taking these with $a = 34$, we can obtain all the integer weights from 35 to 40.

Keywords: Reasoning, equations, combinations, exponents

Similarly, we can obtain the integer weights from 33 down to 28. But the integer weights from 7 to 27, both inclusive, cannot be measured.

Returning to the general case, if we are to eliminate the above-mentioned gap, we should make $(2a - 40)$ the successor of $b + c + d$, i.e., the successor of $40 - a$. That is, we must have $40 - a + 1 = 2a - 40$ or, $3a = 81$ or, $a = 27$.

By similar arguments, we find that $b = 9$, $c = 3$, $d = 1$.

Now, $1 + 3 + 9 + 27 = 3^0 + 3^1 + 3^2 + 3^3 = \frac{(3^4-1)}{3-1} = 40$.

The situation can be extended by adding subsequent powers of 3. That is, by including a standard weight of 81 pounds, one could measure all integer pound weights from 1 to 121 $[=\frac{(3^5-1)}{3-1}]$, and so on.

We now ask: In how many ways can four different weights be distributed in two pans of a balance if the two pans are indistinguishable (that is, they are not labelled A or B or left or right or in any other way)? We list the possibilities below. The possibility of not placing any weight on either pan is omitted.

Distribution of weights	Number of ways
All four in one pan	1
Three in one pan, one in the other	4
Three in one pan, none in the other	4
Two in each pan	3
Two in one pan, one in the other	12
Two in one pan, none in the other	6
One in each pan	6
One in one pan, none in the other	4

These give a total of 40 combinations. So, each integer pound weight from 1 to 40 can be obtained with the weights 1, 3, 9 and 27 pounds in a unique manner.

If the pans are designated in some way (say, left/right), then the number of possibilities doubles to 80. And if we include the possibility of not placing any weight on either pan, then we have 81 possibilities. This can be visualised in another way. Each of the weights can go into the left or right pan or be put aside: 3 possible outcomes. So, with four different weights we have $3^4 = 81$ possibilities.

A. RAMACHANDRAN has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.