# An Easy Proof of Ptolemy's Theorem

RADHAKRISHNAMURTY PADYALA

Many proofs are available ([1], [2], [3]) for the famous and important theorem in geometry known as Ptolemy's theorem. For our discussion, we consider the proof presented by Shirali ([1]). In his article, he described a simple geometrical proof of the theorem and presented two elegant applications. He noted that the proof 'presents a challenge' because from the statement of the theorem we get no clue on how to tackle it. In the proof, there arises a crucial idea of locating a point $E$ on a diagonal of the quadrilateral that enables the construction of two similar triangles. A recent demonstration by Tunsteno [2] demonstrates a simple and intuitively appealing method for locating the point $E$. We present it here for the benefit of school students and teachers.

## Statement of Ptolemy's theorem

**Theorem.** If $ABCD$ is a cyclic quadrilateral, then we have the following equality:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \qquad (1)$$

In words: "The sum of the products of opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals" (see Figure 1).

The crucial idea in the proof described by Shirali is to locate a point $E$ on diagonal $AC$ such that $\triangle CDE \sim \triangle BDA$ (this amounts to $\measuredangle CDE = \measuredangle BDA$; see Figure 2).

*Keywords: Ptolemy's theorem, cyclic quadrilateral, rotation, similar triangles, proof*

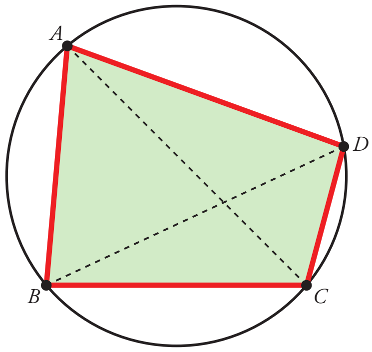
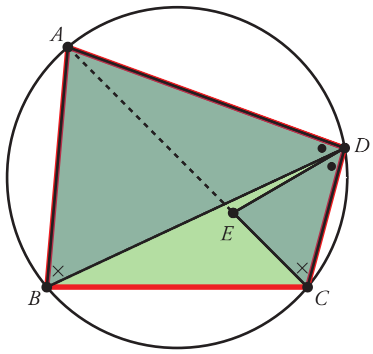



Figure 1. Cyclic quadrilateral $ABCD$ and Ptolemy's theorem



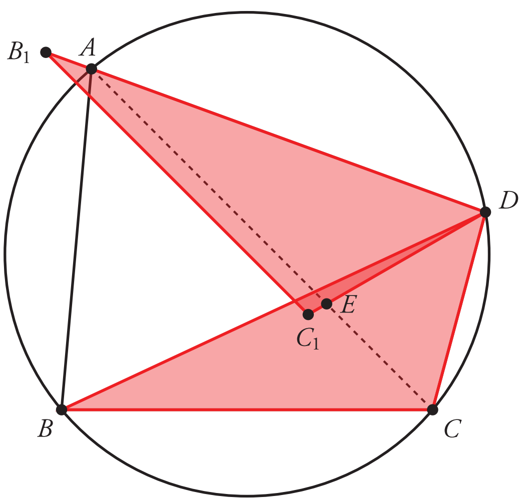

Locate point $E$ on $AC$ such that $\measuredangle CDE = \measuredangle BDA$. Then $\triangle CDE \sim \triangle BDA$.

Figure 2. Point $E$ is located on diagonal $AC$ such that $\measuredangle CDE = \measuredangle BDA$

**Tungsteno's method of locating the point $E$**

The idea proposed by Tungsteno [2] is ingenious. We rotate $\triangle BDC$ around vertex $D$ as pivot (clockwise) through $\measuredangle ADB$, so that the image of side $DB$ lies along line $DA$. Let the image of the triangle be $\triangle B_1DC_1$ (see Figure 3). Let $C_1D$ intersect diagonal $AC$ at $E$. Since $\measuredangle B_1DC_1 = \measuredangle BDC$, it follows that $\measuredangle C_1DC = \measuredangle ADB$. That is, $\measuredangle EDC = \measuredangle ADB$.

Similarly, we rotate $\triangle ADB$ around vertex $D$ as pivot (counterclockwise this time) through $\measuredangle BDC$, so that the image of side $DB$ lies along line




$$\measuredangle DAC = \measuredangle DBC$$

$$\measuredangle DBC = \measuredangle DB_1C_1$$

$$\therefore\ \measuredangle DAC = \measuredangle DB_1C_1$$

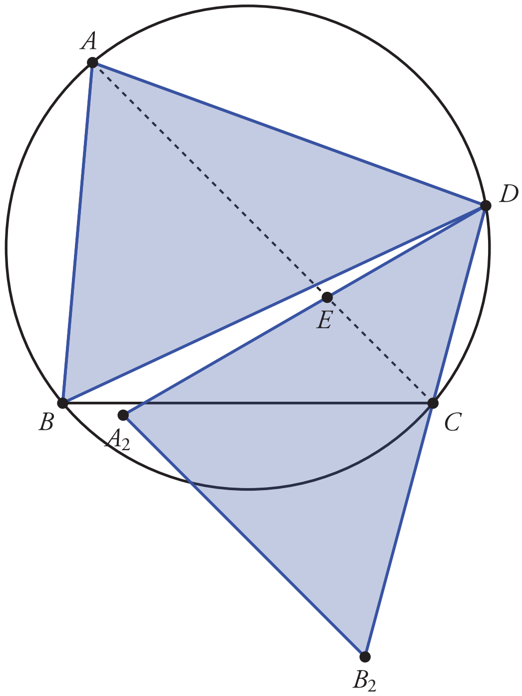
$$\therefore\ AE \parallel B_1C_1$$

$$\therefore\ \triangle DAE \sim \triangle DB_1C_1$$

$$\therefore\ AE : B_1C_1 = DA : DB_1$$

$$\therefore\ AE = \frac{BC \cdot AD}{BD}$$

Figure 3. $\triangle DB_1C_1$ is the image of $\triangle DBC$ under a clockwise rotation about $D$ through $\measuredangle ADB$




$$\measuredangle DCA = \measuredangle DBA$$
$$\measuredangle DBA = \measuredangle DB_2A_2$$
$$\therefore\ \measuredangle DCE = \measuredangle DB_2A_2$$
$$\therefore\ EC \parallel A_2B_2$$
$$\therefore\ \triangle DEC \sim \triangle DA_2B_2$$
$$\therefore\ EC : A_2B_2 = DC : DB_2$$
$$\therefore\ EC = \frac{AB \cdot CD}{BD}$$

Figure 4. $\triangle DA_2B_2$ is the image of $\triangle DAB$ under a counter-clockwise rotation about $D$ through $\measuredangle BDC$

$DC$. Let the image of the triangle be $\triangle A_2DB_2$ (see Figure 4). Since $\measuredangle A_2DB_2 = \measuredangle ADB = \measuredangle C_1DC = \measuredangle EDC$, it follows that side $A_2D$ lies along side $C_1D$, so the intersection of $A_2D$ with diagonal $AC$ yields the very same point $E$ as earlier.

From the derivations shown in Figure 3 and Figure 4, we get:

$$AE = \frac{BC \cdot AD}{BD}, \qquad EC = \frac{AB \cdot CD}{BD},$$

and therefore by addition,

$$AC = \frac{BC \cdot AD}{BD} + \frac{AB \cdot CD}{BD},$$
$$\therefore\ AC \cdot BD = BC \cdot AD + AB \cdot CD. \qquad (2)$$

Thus we prove Ptolemy's theorem easily by this method. □

## References


1. Shailesh Shirali, "How to Prove it: Ptolemy's Theorem", *At Right Angles*, Vol 5, Nov 2016.
2. https://twitter.com/74WTungsteno/status/1373998345497219075
3. A Miraculous Proof (Ptolemy's Theorem) - Numberphile, https://www.youtube.com/watch?v=bJOuzqu3MUQ


**DR. RADHAKRISHNAMURTY PADYALA** is a retired scientist from CECRI (CSIR). He developed an interest in mathematics by reading Martin Gardner's 'Recreational Mathematics' column in *Scientific American*. At present he operates as a freelancer. He has a particular interest in analysing fallacies arising from the incorrect application of mathematics in natural phenomena. He is an ardent admirer of the works of Galileo and Ptolemy. His specialisations are Electrochemistry, Classical Thermodynamics and Kinematics. He may be contacted at padyala1941@yahoo.com.