Middle School Problems
Theme: Number of Digits

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In carrying out operations such as multiplication, division, squaring, cubing, finding square roots, finding cube roots, etc., with natural numbers, we may sometimes be interested in knowing the number of digits in the solution before obtaining the precise result. Often this can help in quickly spotting errors. This exercise aims to raise awareness in this area.

1. A two-digit, a three-digit and a five-digit number are multiplied together. How many digits can the product have?

2. A three-digit number is raised to the power of two. How many digits can the resulting number have?

3. A four-digit number is raised to the power of three. How many digits can the resulting number have?

4. This question takes up the reverse of the situations of questions 2 and 3. If the fourth power of a natural number N has 25 digits, how many digits does N have?

5. Referring to Problem 2, we now ask: What is the dividing line between three-digit numbers with five-digit squares and those with six-digit squares? Or, what is the largest three-digit number whose square has five digits? (Equivalently, what is the least three-digit number with a six-digit square?)

6. A similar question about the cube of a four-digit number: What is the largest four-digit number whose cube has (a) ten digits (b) eleven digits?

Solutions

1. The smallest two-digit number is 10 ; the least three-digit number is 100 ; the least five-digit number is 10000. The product of these numbers is 10000000, which has eight digits. So this is the least possible number of digits in the product. The greatest corresponding numbers are (respectively) one less than 100, 1000 and 100000, whose product (10000000000) has eleven digits. So the product of the greatest two-digit, three-digit and five-digit numbers must have ten digits. So the product could have eight, nine or ten digits.

2. The smallest three-digit number is 100, whose square (10000) is a five-digit number. The greatest three-digit number is one less than 1000, whose square (1000000) has seven digits. So the maximum number of digits that the square of a three-digit number can have is six. That is, the square of a three-digit number can have five or six digits.

3. The smallest four-digit number is 1000, whose cube has ten digits. The greatest four-digit number is one less than 10000, whose cube has thirteen digits. Thus we can see that the cube of the greatest four-digit number would have twelve digits. So the cube of a four-digit number can have ten, eleven or twelve digits.

4. By now you may be able to see that the fourth power of a natural number of d digits can have $4d$, $4d-1$, $4d-2$ or $4d-3$ digits. So one of these expressions must be equal to 25. Obviously, $4d-3=25$ is the only permissible relation (as the others lead to values of d which are not integral); it leads to $d=7$.

   (In the case of fourth powers we could take up another line of argument. The square root of a 25-digit perfect square must have 13 digits, and the square root of a 13-digit perfect square must have 7 digits.)

5. If n is a three-digit number whose square has five digits and $(n+1)$ is a three-digit number with a six-digit square, then n^2 is at most 99999, so n^2 is less than 100000 ; and $(n+1)^2$ is at least 100000.

   Taking square roots of all these quantities, we see that n is less than $\sqrt{10}\times 100$, while $(n+1)$ is at least equal to $\sqrt{10}\times 100$. So $n < 316.2$ and $316.2 \le n+1$. Since n and $(n+1)$ differ by 1, we get $n=316$ and $(n+1)=317$. That is, 10^5 lies between 316^2 and 317^2.

6. Let's look at the two cases separately.

   a. If n is a four-digit number with a ten-digit cube, and $(n+1)$ is a four-digit number with a eleven-digit cube, then n^3 is less than 10000000000, and $(n+1)^3$ is at least 10000000000.

      Taking cube roots of all the quantities, we see that n is less than $\sqrt[3]{10}\times 1000$, while $(n+1)$ is at least equal to $\sqrt[3]{10}\times 1000$. This works out as $n < 2154.4$ and $2154.4 \le n+1$. This means that $n=2154$ and $(n+1)=2155$. That is, 10^{10} lies between 2154^3 and 2155^3.

   b. If n is a four-digit number with a eleven-digit cube and $(n+1)$ is a four-digit number with a twelve-digit cube, then $n^3 < 100000000000 \le (n+1)^3$ giving $n < \left(\sqrt[3]{10}\right)^2 \times 1000 \le (n+1)$. This works out as $n < 4641.6 \le n+1$. Thus, $n=4641$ and $(n+1)=4642$. That is, $4641^3 < 10^{11} < 4642^3$.

   You should be able to obtain the square and cube roots of 10 from the internet.