Solution to a Problem from the CMI Entrance Test 2020

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Problem (CMI entrance paper 2020)**.** Suppose A, B, C, D are points on a circle such that AC and BD are diameters of the circle. Suppose $AB = 12$ and $BC = 5$. Let P be a point on the arc of the circle from A to B (the arc that does not contain the points C and D). Let the distances of P from A, B, C, D be a, b, c, d respectively. Find the values of $\frac{a+b}{c+d}$ and $\frac{a-b}{d-c}$. You may assume $d \neq c$ so the second ratio makes sense.

Solution using Ptolemy's theorem. As AC and BD are diameters, $\measuredangle ABC = 90°$. As $AB = 12$, $BC = 5$, we get $AC = BD = 13$. Also, $\measuredangle BAD = \measuredangle ADC = \measuredangle BCD = 90°$, and $ABCD$ is a rectangle (see Figure 1).

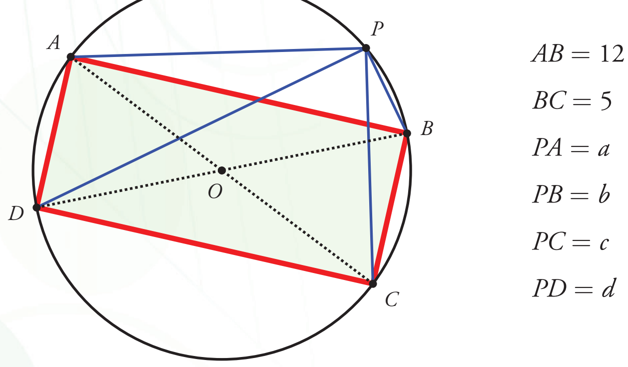

Figure 1.

Keywords: Ptolemy's Theorem, cyclic quadrilateral, properties of a circle, trigonometry

Since $\measuredangle APC = 90^\circ = \measuredangle BPD$, we get:

$$a^2 + c^2 = 13^2, \tag{1}$$

$$b^2 + d^2 = 13^2. \tag{2}$$

From (1) and (2), we get:

$$\begin{aligned} &a^2 - b^2 + c^2 - d^2 = 0, \\ \therefore\ &a^2 - b^2 = d^2 - c^2, \\ \therefore\ &(a+b)(a-b) = (c+d)(d-c), \\ \therefore\ &\frac{a+b}{c+d} = \frac{d-c}{a-b}. \end{aligned} \tag{3}$$

Now apply Ptolemy's theorem (for the statement of the theorem, please refer to the appendix at the end of the article) in turn to the cyclic quadrilaterals *APBC*, *APBD*, *APCD* and *PBCD*:

$$\begin{aligned} 12c &= 13b + 5a, \qquad \therefore\ 13b = 12c - 5a, \\ 12d &= 13a + 5b, \qquad \therefore\ 13a = 12d - 5b, \\ 13d &= 12a + 5c, \\ 13c &= 12b + 5d. \end{aligned}$$

By adding the above we get:

$$\begin{aligned} 13(a+b+c+d) &= 12(a+b+c+d) + 5(c+d-a-b), \\ \therefore\ 6(a+b) &= 4(c+d), \\ \therefore\ \frac{a+b}{c+d} &= \frac{2}{3}. \end{aligned} \tag{4}$$

Recalling (3), we get:

$$\frac{a-b}{d-c} = \frac{3}{2}. \tag{5}$$

Solution using trigonometry. Let the given circle be regarded as the unit circle, with its centre at the origin. Rotating the figure as needed (see Figure 2), we may suppose that $B = (1, 0)$ and $D = (-1, 0)$.

Let $\measuredangle POB = x$ and $\measuredangle AOB = y$. (Here $0 < x < y$.) The coordinates of P, A, C are, respectively, $P = (\cos x, \sin x)$, $A = (\cos y, \sin y)$, $C = (-\cos y, -\sin y)$.

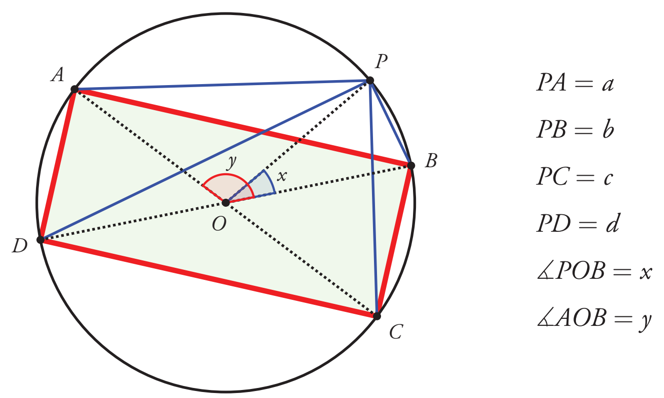

Figure 2.

Using the cosine rule, we get:

$$a = PA = \sqrt{2 - 2\cos(y-x)} = 2\sin\frac{y-x}{2}, \tag{6}$$

$$b = PB = \sqrt{2 - 2\cos(x)} = 2\sin\frac{x}{2}, \tag{7}$$

$$c = PC = \sqrt{2 + 2\cos(y-x)} = 2\cos\frac{y-x}{2}, \tag{8}$$

$$d = PD = \sqrt{2 + 2\cos(x)} = 2\cos\frac{x}{2}. \tag{9}$$

Therefore:

$$\frac{a+b}{c+d} = \frac{\sin\frac{y-x}{2} + \sin\frac{x}{2}}{\cos\frac{y-x}{2} + \cos\frac{x}{2}} = \frac{2\sin\frac{y}{4}}{2\cos\frac{y}{4}} = \tan\frac{y}{4}. \tag{10}$$

From (10) we see that the required ratio is independent of the position of the point P.

So we opt for the most convenient choice and let $P = A$. For this choice of P, we have $a = 0$, $b = AB = 12$, $c = AC = 13$, $d = 5$. Hence:

$$\frac{a+b}{c+d} = \frac{12}{18} = \frac{2}{3}, \tag{11}$$

and a similar calculation yields (or we may use (3) and get the same result):

$$\frac{a-b}{d-c} = \cot\frac{y}{4} = \frac{3}{2}. \tag{12}$$

Appendix: Ptolemy's theorem. The theorem states that if $ABCD$ is a cyclic quadrilateral, then the following equality holds:

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

That is, *the sum of the products of opposite pairs of sides is equal to the product of the diagonals.*

References

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