A Functional Equation from APMO 2019

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In this article, we give a solution to Problem 1 from the Asia Pacific Mathematics Olympiad 2019:

Problem 1. Let $\mathbb{Z}^+$ be the set of positive integers. Determine all functions $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$ for all positive integers a and b.

Notation. If m and n are integers ($m \neq 0$), the notation $m \mid n$ means that m is a divisor of n. For example, $4 \mid 12$.

Solution. We use mathematical induction and show that $f(x) = x$ is the only function satisfying the given functional equation.

Let $P(a, b)$ denote the statement that $a^2 + f(a)f(b)$ is divisible by $f(a) + b$. We argue as follows.

- $P(1, 1)$ tells us:

$$\begin{aligned} & f(1) + 1 \mid f(1)^2 + 1, \\ \therefore\ & f(1) + 1 \mid f(1)^2 + 1 - (f(1) + 1), \\ \therefore\ & f(1) + 1 \mid f(1)^2 - f(1), \\ \therefore\ & f(1) + 1 \mid f(1) \cdot (f(1) - 1). \end{aligned}$$

- Since $f(1)$ and $f(1) + 1$ are coprime, $f(1) + 1 \mid f(1) - 1$. This implies that $f(1) + 1 \mid 2$, hence $f(1) = 1$.

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- Let the induction hypothesis be: For some positive integer $k > 1$, the relation $f(x) = x$ is true for $x = 1, 2, 3, \ldots, k-1$. We shall now prove that $f(k) = k$.
- $P(k-1, k)$ tells us (since $f(x) = x$ for $x < k$, by assumption):

  $$2k-1 \mid (k-1)^2 + (k-1)f(k),$$
  $$\therefore\ 2k-1 \mid (k-1)\cdot(k-1+f(k)).$$

  Since $\gcd(2k-1, k-1) = \gcd(2k-1, 2k-2) = 1$, it follows that

  $$2k-1 \mid k-1+f(k). \tag{1}$$

- Next, $P(k, k-1)$ tells us:

  $$f(k)+k-1 \mid k^2 + (k-1)f(k),$$
  $$\therefore\ f(k)+k-1 \mid k^2 + (k-1)f(k) - (k-1)\cdot(f(k)+k-1),$$
  $$\therefore\ f(k)+k-1 \mid k^2 - (k-1)^2,$$
  $$\therefore\ f(k)+k-1 \mid 2k-1. \tag{2}$$

- From (1) and (2) we get $f(k) + (k-1) = 2k-1$, so $f(k) = k$. We have thus completed the induction step.

It follows that the only function satisfying the given functional equation is $f(x) = x$. $\square$

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