Review: Counters

Reviewed by Math Space

Simply put, anything that can be counted may be considered to be counters. So, pebbles, chalk pieces, bottle caps, seeds, buttons – all these are familiar objects which can be utilised as counters. Square counters can be made easily from (corrugated) cardboard boxes by drawing a square grid with squares of side at least 2cm and then cutting it up. Cutters (with steel scales) work much faster for corrugated cardboard than scissors. Making round counters from cardboard is more tedious but not impossible (with difficulty decreasing as size increases). Carrom coins, bottle caps (ideally of the same size), tooth paste caps are great as round counters. It is advisable to paint the counters in bright colours especially for younger children.

Counters can help a child establish a one-to-one correspondence between the quantity (count) and the numeral or number name that represents that quantity (Figure 1). They are the first manipulatives that any child should encounter. Counters can help one focus on the quantity and not how spread apart they are (Figure 2). As manipulatives, counters can be picked up (and set apart in a bowl or basket) – thereby making it easier to separate out what has been counted from what is yet to be counted. In fact, one may say that without adequate practice of counting, a child will not be able to have a strong foundation of numbers. Therefore, counters are very crucial at the beginning.

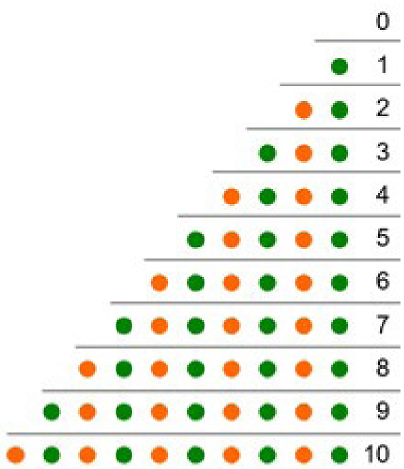


Figure 1

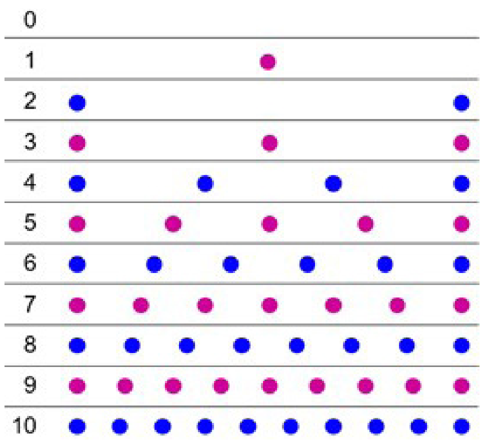


Figure 2

However, the advantages of counters are not limited to just counting. They help equally in terms of identifying any pattern. Therefore, they also provide a great starting point for algebraic thinking and algebra in general. Growing (Figure 3) and repeating (Figure 4) patterns can initiate a discussion around algebraic expressions and equations.

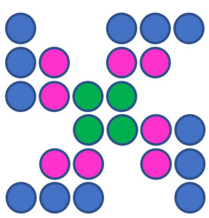


Figure 3

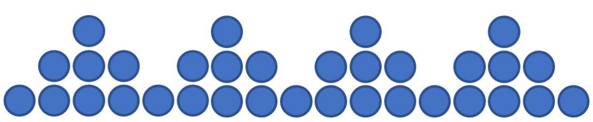


Figure 4

Moreover, they illustrate properties of addition and multiplication beautifully for discrete number sets, viz., whole numbers and integers. Here are some examples:

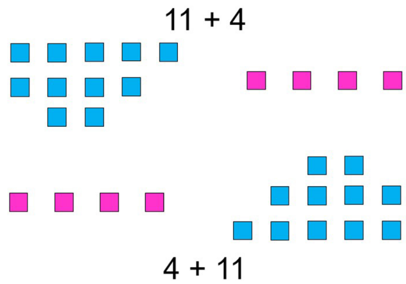


Figure 5

Figure 5 depicts the same sum from two perspectives while Figure 6 illustrates that no matter which two groups are combined first, the sum remains the same (see reference [2] for further details). For integers, since there are two types of numbers – positive and negative, we need two types of counters. Basically, the number of counters indicates the distance of the integer from zero, i.e., the absolute value and the colour (or type) indicates which side of zero it is on, i.e., the sign. Figure 7 and Figure 8 illustrates this with 3 and -2, respectively. Figure 5 and Figure 6 can be extended in a similar manner for integers with such coloured counters (see reference [3] for further details).

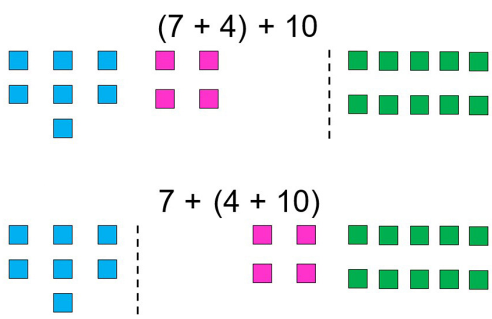


Figure 6

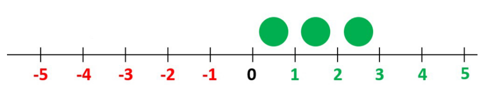


Figure 7

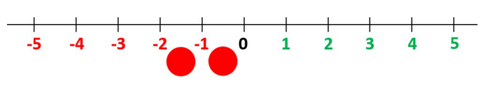


Figure 8

Figure 9 illustrates commutativity of multiplication for whole numbers. The same array depicts two products on changing its orientation. Figure 10 and Figure 11 on the other hand show the distributive property for whole numbers and for integers, respectively (see references [2] and [3] for further details).

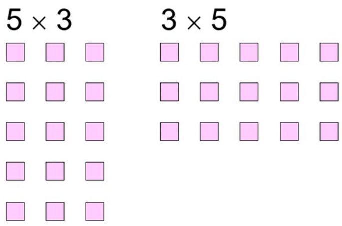


Figure 9

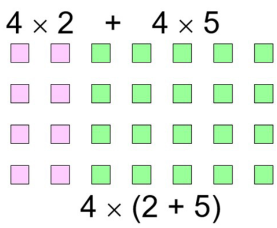


Figure 10

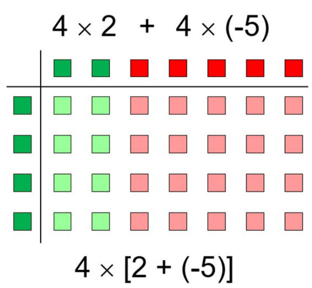


Figure 11

They also help in finding factors, characterising the primes including why 1 is not a prime, and understanding square numbers! Figure 12 depicts all possible arrays with 12 counters. Note that the number of rows (or columns) in each array maps to a factor of 12 and thus the number of possible arrays equals the number of factors.

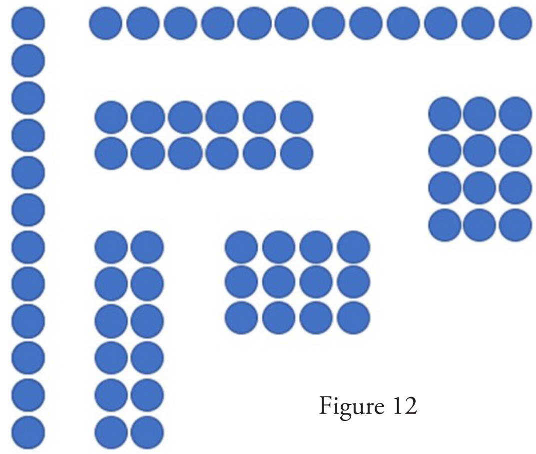


Figure 12

Note how the arrays are portrait-landscape pairs. So, what happens for squares?

<table>
<tr><th>Number</th><th>No. of arrays</th><th>Number</th><th>No. of arrays</th><th>Observations</th></tr>
<tr><td>1</td><td>1</td><td>9</td><td>3</td><td rowspan="8">• Which numbers have exactly 2 possible arrays?
• Which numbers have more than 2 arrays?
• So, is 1 a prime? Is it composite?
• Which numbers have odd number of arrays?</td></tr>
<tr><td>2</td><td>2</td><td>10</td><td>4</td></tr>
<tr><td>3</td><td>2</td><td>11</td><td>2</td></tr>
<tr><td>4</td><td>3</td><td>12</td><td>6</td></tr>
<tr><td>5</td><td>2</td><td>13</td><td>2</td></tr>
<tr><td>6</td><td>4</td><td>14</td><td>4</td></tr>
<tr><td>7</td><td>2</td><td>15</td><td>4</td></tr>
<tr><td>8</td><td>4</td><td>16</td><td>5</td></tr>
</table>

The other proof of 'why square numbers have odd number of factors' involves the Fundamental Theorem of Arithmetic (or the unique prime factorization theorem) and uses algebra (and is therefore much more demanding)!

The observation column scaffolds the path to abstraction of concepts as students begin to extend their thinking to pictorial representations and then mental images.

In addition, counters are great for generating Proofs Without Words (PWW) for results involving natural numbers. A resource group has in fact made a model of Gauss's famous childhood discovery of sum of natural numbers! Figure 13 shows the series of odd numbers adding up to a square. In fact, it should be possible to create PWW for every result involving natural numbers using counters!

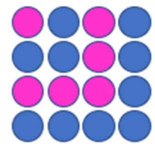
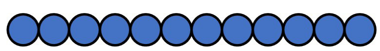


Figure 13

When we go into such higher and exciting levels of math, it helps if the counters are identical. There are two popular shapes – circle and square – each with their own pros and cons.

	Circle	**Square**
Pros	Most symmetric shape ∴ independent of orientation ⇒ best for very young children ⇒ best for patterns since it gives maximum freedom Also, when lined up, they are easier to count than square counters because of the gap between counters	A regular shape that tiles ∴ great to generate anything based on the square grid, e.g., polyominoes Unit of area ∴ great to explore area and perimeter ⇒ when lined up forms a rectangle whose area is proportionate to its length ∴ can be generalized to length
Cons	Does not tile	When arranged in a line forming a rectangle, it may be difficult to count them unless the borders are prominent Unless arranged properly, they may not look nice or neat Less symmetry compared to circles ∴ certain symmetry may not be clear if square tiles are used instead of circular ones
Picture		

References

[1] Multiplication Pullout, *At Right Angles* (Mar 2014)

[2] Exploring Properties of Addition with Whole Numbers and Fractions, *At Right Angles* (Jul 2018)

[3] Exploring Properties of Addition and Multiplication with Integers, *At Right Angles* (Jul 2019)

[4] Nelson, Roger B: *Proof Without Words* (Volume 1)

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] both in terms of uses and regarding possibility of low-cost versions that can be made from boxes, etc. It tries to address both fear and dislike for mathematics as well as provide food for thought to those who like or love the subject. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space may be reached at mathspace@apu.edu.in