techspace

Connections between Geometry and Number Theory

In this article we explore connections between specific numbers and geometry, revealing new connections that you, dear reader, might have overlooked or not have seen before. We do the explorations by using modern technology, illuminating the strength of technology when working with mathematical investigations.

Thomas Lingefjärd

Some oddities of the number 7

A well-known fact is that a week has 7 days. "In six days God made the heaven and the earth, the sea, and all that is in them, but He rested the seventh-day. Therefore the Lord blessed the Sabbath day and made it holy." So the creation story points out 7 as a special number.

The Egyptians had seven original and higher gods; the Phœnicians had seven kabiris; the Persians had seven sacred horses of Mithra; the Parsees seven angels opposed by seven demons, and seven celestial abodes paralleled by seven lower regions. The seven gods were often represented as one seven-headed deity. The whole heaven was subjected to the seven planets; hence, in nearly all the religious systems we find seven heavens.

An important cognitive ability within humans is memory span. Memory span often refers to the longest possible list of items (e.g., colours, digits, letters, words) which a person can repeat immediately after a presentation, in the correct order. Millar (1956) has shown that the memory span of humans often is approximately 7 ± 2 items.

According to the theory of biorhythms, a person's life is affected by rhythmic biological cycles which affect one's ability in various domains, such as mental, physical, and emotional activity. These cycles begin at birth and oscillate in a steady sine wave fashion throughout life; by modelling them mathematically, a person's level of ability in each of these domains can be predicted approximately from day to day.
The emotional biorhythm model is a 28-day cycle. Here too the number 7 plays a role.

Mathematically interesting connections

The number 7 is prime, and Archimedes discovered its approximate kinship to the circle. He realized that a circle's circumference can be bounded from below and from above by inscribing and circumscribing regular polygons and computing the perimeters of the inner and outer polygons. By so doing, he proved that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

The first prime which is not 1 more than a power of 2 is 7: thus, $2=2^0+1$, $3=2^1+1$, $5=2^2+1$, but $7=2^3-1$.

A regular polygon with 7 sides is the first regular polygon which cannot be constructed by traditional Euclidean methods using straightedge and compass alone. (After 7 the next two such numbers are 9 and 11.)

The repeating portion of the decimal fraction corresponding to 1/7 is 142857 (that is, 1/7 equals 0.142857 142857 …). We have furthermore that:

142857 x 1	=	142857
142857 x 2	=	285714
142857 x 3	=	428571
142857 x 4	=	571428
142857 x 5	=	714285
142857 x 6	=	857142

The same figures come back in different order! We also see that we can express 1/7 as a geometric sum defined as

$$a\sum_{n=0}^{\infty} k^n = \frac{a}{1-k}$$

where $a = 0.14$ and $k = 0.02$. (The sum evaluates to 0.14/0.98 which simplifies to 1/7.)

Remember the ancient Egyptian and Archimedes approximation for π through 22/7 = 21/7 + 1/7 = 3.142857142857…

Given an integer k, a positive integer x is said to be *k-transportable* if, when its left most digit is moved to the units place (i.e., 'left to right'), the resulting integer is kx.

The integer 142857 is 3-transportable since

$$428571 = 3\times142857.$$

Kahan (1976) proved that for $k > 1$ there are no such integers unless $k = 3$, and the 3-transportable integers all belong to one of the following two sequences:

$$142857,\ 142857142857,\ 142857142857142857,\ \ldots$$

$$285714,\ 285714285714,\ 285714285714285714,\ \ldots$$

The following strange connection between algebra, geometry and the fraction 1/7 was shown to the author by the Swedish mathematician Andrejs Dunkels in 1988. Dunkels challenged us to show that if we combine six overlapping pairs of digits in 142857, and thereby get the following Cartesian points in the plane (1, 4), (4, 2), (2, 8), (8, 5), (5, 7) and (7, 1), these six points lie on an ellipse.

This astonishing fact was first pointed out in 1986 by Edward Kitchen, who encouraged readers of *Mathematics Magazine* (problem section) to prove the fact noted above. See Figure 1 constructed with GeoGebra. The problem is easily solved by Dynamical Geometry software (e.g., GeoGebra or Geometer's Sketchpad), but in the October 1987 issue of the magazine the problem was solved by hand by John C. Nichols, Thiel College, Pennsylvania.

It is well known that five arbitrary points satisfy a conic equation given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We have six coefficients to determine; but they are determined up to multiplication by a non-zero constant (that is, if the six numbers are scaled up by a common constant, we get the same conic), which means that five points determine the conic (provided that no four of them lie on a line; if three of the points lie on a line, the conic is a union of two lines).

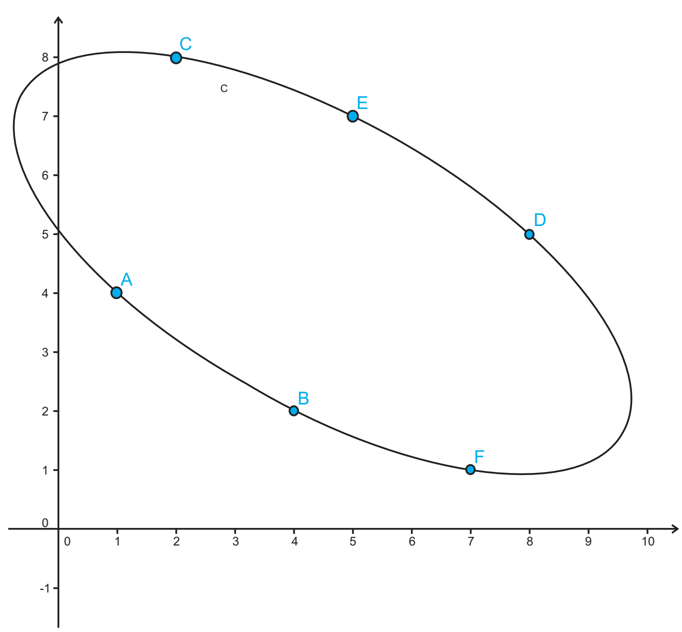

Figure 1: The 1/7 ellipse, where A = (1, 4), B= (4, 2), C = (2, 8), D = (8, 5), E = (5, 7), F = (7, 1).
Its equation is: $19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0$.

The fact that in the ‘1/7 ellipse’ the sixth point too lies on the conic rests on a symmetric relation that holds between the six points; specifically, on the fact that 142 + 857 = 999, which yields the following relation (the significance of this will be seen presently):

$$(1,4) + (8,5) = (4,2) + (5,7) = (2,8) + (7,1) = (9,9).$$

What other reciprocals have the same qualities? What will for instance happen if we combine the points (14, 28), (28, 57), (57, 14) with the points (42, 85), (85, 71), (71, 42)? (These are obtained by taking 2-digit combinations from the decimal expansion of 1/7.) It happens that these six points too lie on an ellipse; see Figure 2.

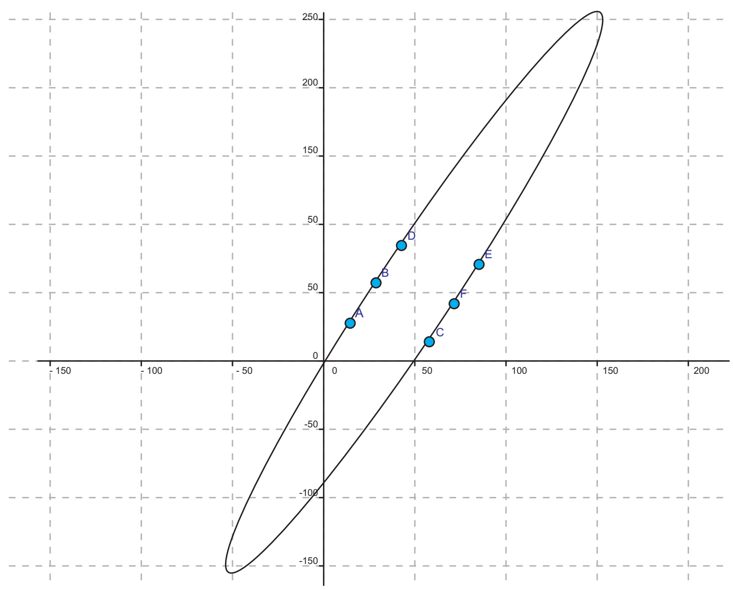

Figure 2: Variant of the 1/7 ellipse,with A = (14, 28), B= (28, 57), C = (57, 14), etc.
Its equation is: $165104\,x^2 - 160804\,x\,y + 41651\,y^2 - 8385498\,x + 3836349\,y + 7999600 = 0$.

Generalizing the question

The Shippensburg University problem solving group (1987) investigated all ‘period six reciprocals’ (i.e., those whose digital forms have a six-digit repetend, like 1/7) and found that reciprocals of 13 and 77 yield hyperbolas, the reciprocals of 39, 63, 91, 143, 273, 429, 693 and 819 yield ellipses, while the reciprocals of 21, 117, 189, 231, 259, 297, 351, 407, 481 and 777 do not yield a conic at all.

Mathpuzzle (December 2006) cited Chris Lomont: “Out of curiosity, I found a lot more of these ellipses. One with more points is the 1/7373 ellipse, 1/7373 = 0.00013653... which gives seven points (0,0), (0,1), (1,3), (3,0), (3,5), (5,6), (6, 3) on an ellipse. To get 8 points on a single ellipse I found that the fraction 4111/3030303 works. I’ve yet to find more on a single ellipse. I'm unaware of any proof that can be done, although integer points on curves are much studied.”(Web reference).

The first 6 pairs of numbers in several decimal fractions lie on an ellipse (e.g. 23/91 or 75/91) or on a hyperbola (e.g. 2/13 or 36/91).

Further, one might investigate the effect of considering for the coordinates not just single digits but blocks of digits of various lengths (2, 3, ...). I found that the blocks of length 2 of several reciprocals including 1/7, 1/13, 1/77, 1/91 and 1/819 yield conics but the blocks of length 2 of 1/7373 (period 8 reciprocal with 7 points on a conic) do not yield a conic. Moreover, blocks of length 3 of the reciprocals 1/7, 1/13, 1/77, 1/91 yield the straight line $y = -x + 999$, whereas blocks of length 3 of 1/819 yield the straight line $y = -x + 222$.

For example blocks of length 2 of 1/13 yield a hyperbola with the equation see (Figure 3; the caption shows how the coordinates of the points are computed):

$$-4013x^2 + 36478xy - 53117y^2 - 1408374x + 3452922y + 7074800 = 0.$$

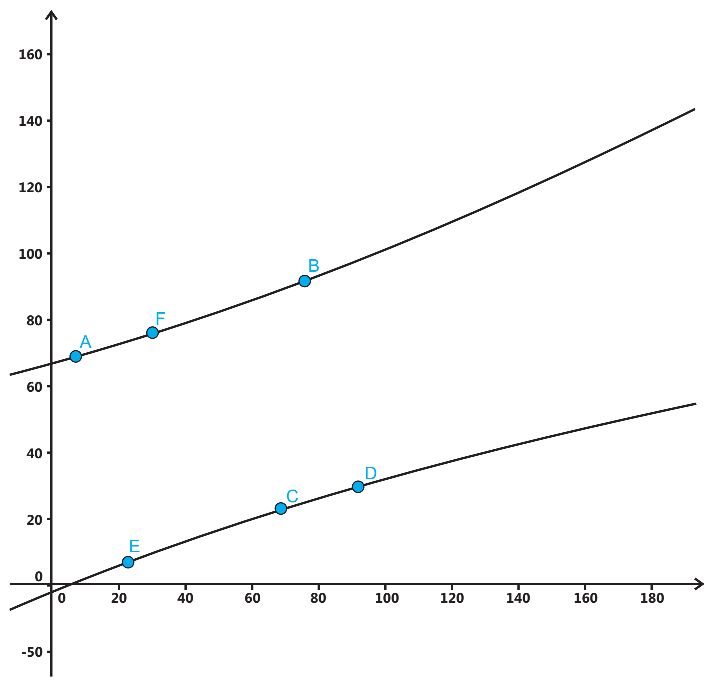

Figure 3: Points produced from blocks of length 2 from 1/13 = 0.076923…, yield a hyperbola. Here, A = (07,69), B = (76,92), C=(69,23), D = (92, 30), E = (23, 07), F = (30, 76)

The centres of the conics of 1/7 and 1/13 are all located at (9/2, 9/2) whereas the centres of the conics connected with blocks of length 2 are located at (99/2, 99/2).

Analysis: One way to look at digital-conics is that if you have four numbers $a, b, c, d,$ then the following six points necessarily lie on a central conic with centre $(d/2, d/2)$:

$$(a, b),\ (b, c),\ (c, d-a),\ (d-a, d-b),\ (d-b, d-c),\ (d-c, a).$$

Which particular conic manifests (hyperbola or ellipse) depends on the values of $a, b, c, d.$ However it seems difficult to make a precise correlation. It would be an interesting project to explore this correlation further.

In the case of 1/7 we have $a = 1$, $b = 4$, $c = 2$, $d = 9$ which as noted earlier draws on the fact that 142 + 857 = 999, and in a similar way the following six points lie on a conic:

$$(14,28), (42,85), (28,57), (85,71), (57,14), (71,42).$$

This can be seen as

$$(a,b),\ (d-c,d-a),\ (b,c),\ (d-a,d-b),\ (c,a),\ (d-b,d-c),$$

with $a = 14$, $b = 28$, $c = 57$, $d = 99$; in this case the coordinates are permuted in two different cycles of length 3 while we in the previous case had one single cycle of length 6. It is worth noting that in both cases the centre of the ellipse lies at $(d/2, d/2)$.

If we go back to the example of the 1/7 ellipse, the ellipse can be described as

$$19(2x-9)^2 + 36(2x-9)(2y-9) + 41(2y-9)^2 = 1224.$$

Also other 6-tuples of numbers can be used. They do not need to be different; for example, the number 112332 with ($a = b = 1$, $c = 2$, $d = 4$) gives 6 points that lie on the ellipse $3\,(x-y)^2 + (x+y-4)^2 = 4$.

If pairs of *triplets* of a period six reciprocal lie on the same line, the slope of the line must be s = −1. This is so because the first and fourth point have the same coordinates but in reverse order: $x_1 = y_4$ and $x_4 = y_1$ which gives $s = -1$.

What happens if we multiply 7 with 13, giving 91? We have: 1/91 =0.010989 ...which yields the points (0, 1),(1, 0), (0, 9), (9,8), (8, 9) and (9, 0). See Figure 4.

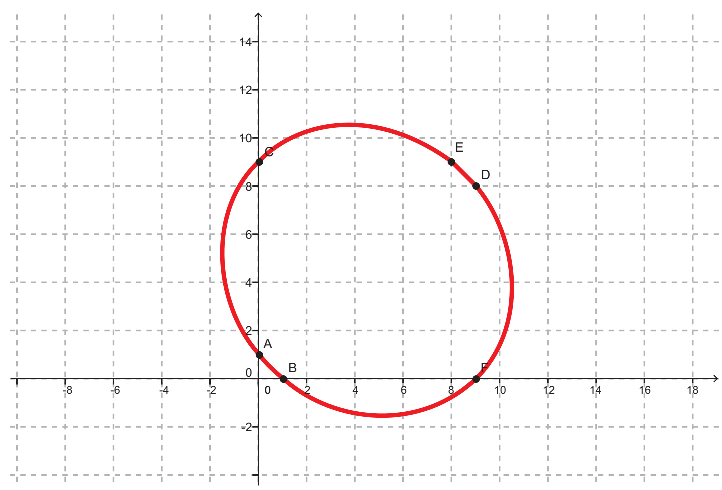

Figure 4: The ellipse built on the fraction 1/91

So far we have worked mainly on numbers generating fractions. What if we ask the question the other way round? For instance, is there a fraction $1/n$ with a cycle of length eight that yields an ellipse? It is not that hard to find that 1 / 73 = 0.01369863013 gives eight points (0, 1), (1, 3), (3, 6), (6, 9), (9, 8), (8, 6), (6, 3), (3, 0). See Figure 5.

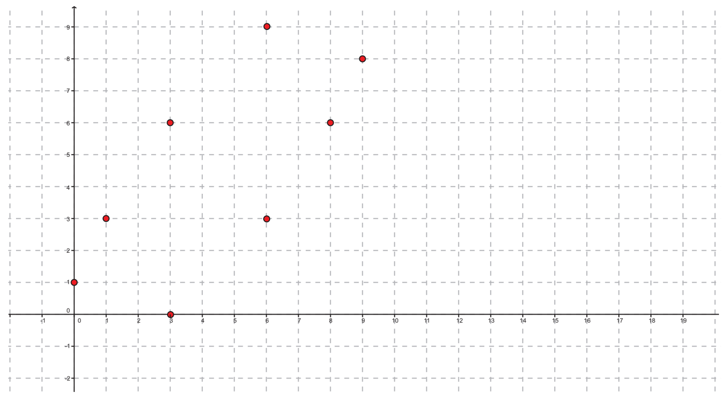

Figure 5: points derived from the fraction 1/73

Once the points are sketched, we see that they lie on an oval with centre (9/2, 9/2). But they do not all lie on an ellipse, as we find if we ask GeoGebra to fit a conic to the points. But if we omit (9, 8) and (0, 1) we do get an ellipse that not only goes through the six remaining points, but also through (0, 0), (1, –1), (8, 10) and (9, 9). We call this a *ten point ellipse.* The equation is
$3(2x-9)^2 + 2(2y-9)^2 - 4(2x-9)(2y-9) = 81$. The midpoint is at (9/2, 9/2). See Figure 6.

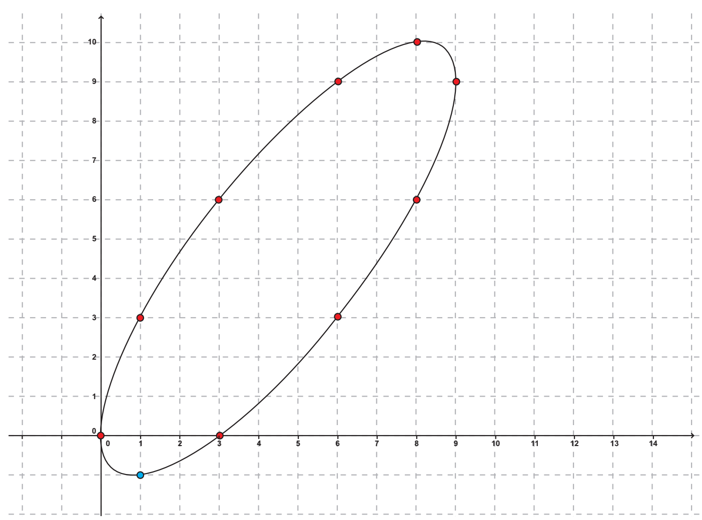

Figure 6: An ellipse partly derived from the fraction 1/73

In Figure 7, you find an ellipse with 18 outspread integer points on the periphery. Isn't that beautiful?

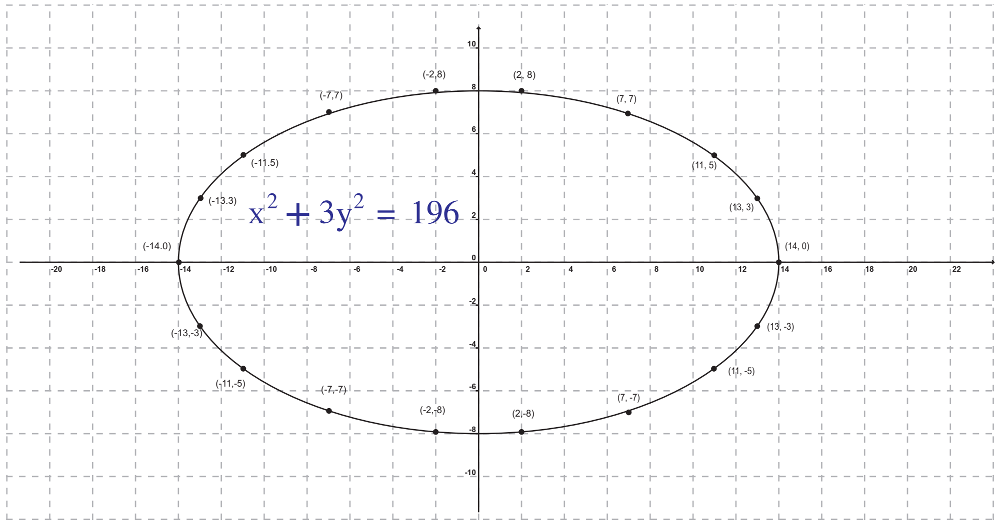

Figure 7: An ellipse with 18 integer points

Obviously we can go in many different directions. I leave further investigations to the reader.

Acknowledgements

I am thankful to Thomas Weibull at Chalmers Technical University, and StureSjöstedt, Sweden, to Jim Wilson, University of Georgia, USA and to Erik Hansen and Peter Limkilde, Odsherreds Gymnasium, Denmark, for helpful comments and advice on a previous (unpublished) version of this article.

Worksheet

I. Investigate reciprocals of positive integers. Which reciprocals have 6 digit repetends? Make a record of these.

II. Focus on one particular reciprocal, say the reciprocal of 7. Here are some sample exercises:

1. Fill in the values for *a,b,c* so as to give a set of points that lie on a straight line:(142, 857), (285, 714), (428, *a*), (*a, b*), (857, *c*). What is its equation?
2. Do the points (1, 4),(4, 2), (2, 8) , (8, 5) and (5, 7) lie on a straight line?

   If they do, what is the equation of the line?

   If not, use Geogebra to investigate if they lie on a conic (remember that 5 points lie on a conic if no set of four points are in a straight line).

   If they do lie on a conic, check whether (7, 1) lies on the conic. Explain your finding.
3. Do the reflections of these points in the line $y = x$ lie on a conic? What is its centre, if so?
4. Substitute values for a, b, c and d and plot the 6 points (a, b), (b, c), $(c, d-a)$, $(d-a, d-b)$, $(d-b, d-c)$, $(d - c, a)$.
   - What happens if $a = b = c$?
   - If a, b, c are distinct, why do these points give a conic centred at $(d/2, d/2)$?
   - Will the same conclusion hold if a, b, c, d are two digit numbers?

References

i. Dunkels, A. (1988). Have I shown it yet? [Har jag visatdet nu?]. Paper presented at the Swedish Mathematical Association's Education day 1988-03-17.

ii. Kahan, S. (1976). *K-transportable integers. Mathematical Magazine* (49) 27 – 38.

iii. Kitchen, E. (1986).A curious property of 1/7. *Mathematics Magazine*, vol. 59, no. 4, p. 245.

iv. Nichols, J., C. (1987). Solution to a curious property of 1/7. *Mathematics Magazine,* vol. 60, no. 4, p. 245.

v. The Shippensburg University Problem Solving Group.(1987). Solution to a curious property of 1/7. *Mathematics Magazine,* vol. 60, no. 4, p. 245.
http://www.mathpuzzle.com/25Dec2006.html

Thomas Lingefjärd is Professor of Mathematics Education at the University of Gothenburg, Sweden, and is deeply interested in both mathematical and educational issues. The impact of ICT on learning is something he has been interested in for 25 years. He visited India in 2011 for the India-Swedish Math Education initiative, and has since then been collaborating and working with Jonaki Ghosh. He may be contacted at thomas.lingefjard@gu.se.

Notes added by the Editors

- The facts that (i) 7 is the smallest prime not of the form '1 more than a power of 2' and (ii) 'a regular polygon with 7 sides is the first regular polygon which cannot be constructed by traditional Euclidean methods' are, surprisingly, connected closely. We know this from the work of Carl F Gauss on the *Fermat primes.* Specifically, the result proved by Gauss is this:

  For odd integers $n \geq 3$, a regular n-sided polygon can be constructed using ruler-and-compass if and only if n is the product of distinct Fermat primes. (A 'Fermat prime' is a prime of the form $2^k + 1$; for example, 3, 5 and 17.)

  We shall elaborate on this connection in a subsequent article.

- All figures in the above article have been produced using GeoGebra. The 'conic' tool of GeoGebra is easy to use, and we urge the reader to explore further. The syntax is this: if *A, B, C, D, E* are given points, then the conic *k* defined by them is produced by the following command: *k* = Conic (*A, B, C, D, E*). The graphical interface can be used as well.

- The claim that the points $P = (a,b)$, $Q = (b, c)$, $R = (c, d - a)$, $S = (d - a, d - b)$, $T = (d - b, d - c)$, $U = (d - c, a)$ lie on a central conic with centre $M = (d/2, d/2)$ follows from the symmetric nature of this set of points. Let Γ be the conic passing through *P, Q, R, S, T*. (This conic is well-defined subject to some mild restrictions on *a, b, c, d.* For example, we should not have $a = b = c$.) Observe that *PS* and *QT* have the same midpoint, *M*. Let *M* be the origin of a new coordinate system, and let the equation of Γ in this system be $f(x, y) = 0$ where *f* is quadratic. Using the fact that in this system *P* and *S* have the origin as mid-point, as also *Q* and *T*, we argue that the coefficients of *x* and *y* in $f(x, y)$must be zero. Hence $f(x, y)$ must have only terms in the second degree (i.e., x^2, y^2, xy). But this implies that if any point lies on the conic, so does the point whose coordinates are the negatives of the first point. Since *R* lies on the conic, this implies that *U* does too. This justifies the claim.