problem corner

Fun Problems

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k-transportable numbers

In the article *Connections between Geometry and Number Theory* (elsewhere in this issue of *At Right Angles*) the author refers to the notion of 'k-transportable numbers' — numbers with the property that if the left-most digit is shifted to the right-most end, then the number thus obtained is k times the original number. He quotes a result due to S. Kahan that the only integral value of k exceeding 1 for which such a number exists is $k = 3$. We prove this result here (our proof is different from Kahan's), and show a surprising way for generating such numbers. A more apt name for such numbers than the one given would be *cyclic numbers*, and we study this more general notion in the next section.

Let $A = \overline{a_1 a_2 a_3 \dots a_n}$ be a k-transportable number, and let $B = \overline{a_2 a_3 \dots a_n a_1}$, where k is a positive integer ($k \neq 1$; of course, $k < 10$). Then we have:

$$B = kA.$$

Construct the following two infinite recurring decimals, whose 'repetends' (i.e., the portions that repeat indefinitely) are the numbers A and B respectively. That is:

$$x = 0.A\,A\,A\ \dots,$$

$$y = 0.B\,B\,B\ \dots.$$

Both are 'pure' recurring decimals. Since $B = kA$, it follows that $y = kx$. Now consider the effect of multiplying x by 10. Noting the 'shift in the decimal point' we see that this yields

$$10x = a_1.B\,B\,B\ \dots,$$

a number with integer part a_1, and recurring portion the same as that of y. Hence we have:

$$10x = a_1 + y.$$

Since $y = kx$ this yields $10x = a_1 + kx$, and solving this for x we get:

$$x = \frac{a_1}{10 - k}.$$

Since x is a pure repeating decimal fraction, this relation puts restrictions on the value of k. Indeed we have $k \neq 2, 4, 5, 6, 8, 9$. We also have $k \neq 1$. So the possibilities for k are just 3, 7. Of these, $k = 7$ yields $1/(10 - k) = 0.333\dots$, which makes A a single digit number; this does not work out. Hence $k = 3$ (this was Kahan's result), and $x = a_1/7$. Since the repetend of $1/7$ is 142857, we see that $A = a_1 \times 142857$, with a_1 chosen appropriately. Remembering that a_1 is also the left-most digit of A, we find that $a_1 = 1$ or 2; only these two choices work. Hence $A = 142857$ or 285714.

Obviously, repeating these blocks of digits will give more numbers with the same property. This justifies the claim made in the article that the only k-transportable integers are the following:

$$142857, \quad 142857142857, \quad 142857142857142857, \quad \dots,$$

$$285714, \quad 285714285714, \quad 285714285714285714, \quad \dots,$$

all of which are k-transportable with $k = 3$.

Cyclic numbers

The same idea can be used to solve the following: *Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is twice the original one*. Denote the number by $A = \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}$ (so it has n digits), and let $B = \overline{a_n a_1 a_2 a_3 \dots a_{n-1}}$; then $B = 2A$. Let x, y be pure recurring decimals defined as follows:

$$\begin{aligned} x &= 0.A\,A\,A\,A\,\dots \\ &= 0.\overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \dots, \\ y &= 0.B\,B\,B\,B\,\dots \\ &= 0.\overline{a_n a_1 a_2 a_3 \dots a_{n-1}}\ \overline{a_n a_1 a_2 a_3 \dots a_{n-1}}\ \dots. \end{aligned}$$

Then $y = 2x$. If we multiply y by 10 we get a pure decimal recurring decimal whose repetend is the same as that of x:

$$\begin{aligned} 10y &= a_n.\overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \dots \\ &= a_n + x. \end{aligned}$$

Since $y = 2x$ this yields: $20x = a_n + x$, and so:

$$x = \frac{a_n}{19}.$$

It therefore remains only to find the repetend of the fraction $1/19$, which we get by simple long division:

$$\frac{1}{19} = 0.\overline{052631578947368421}.$$

If we choose $a_n = 1$ we get $A = 052631578947368421$, which has 0 as its first digit; so we discard this solution. If we choose $a_n = 2$ we get $A = 105263157894736842$, and we have a possible answer:

$$A = 105263157894736842.$$

Please check that $105263157894736842 \times 2 = 210526315789473684$.

This means that 105263157894736842 *is the smallest possible solution to the problem*.

Other choices for a_n yield more solutions, all using the same repetend. Thus:

$$\begin{aligned} a_n &= 2 \text{ yields } A = 105263157894736842, \\ a_n &= 3 \text{ yields } A = 157894736842105263, \\ a_n &= 4 \text{ yields } A = 210526315789473684, \\ a_n &= 5 \text{ yields } A = 263157894736842105, \\ a_n &= 6 \text{ yields } A = 315789473684210526, \\ a_n &= 7 \text{ yields } A = 368421052631578947, \\ a_n &= 8 \text{ yields } A = 421052631578947368, \\ a_n &= 9 \text{ yields } A = 473684210526315789. \end{aligned}$$

That's a lot of solutions!

Given a positive integer N, let $f(N)$ denote the integer obtained by shifting its units digit to its left-most end. (Example: $f(1234) = 4123$.) A number N with the property that the ratio $f(N) : N$ is an integer, or a rational number with small numerator and denominator, is called a ***cyclic number***. The best known example of such a number is 142857 (for which the ratio is $5 : 1$). Such numbers are always associated with the repetends of pure recurring decimals (and that is what helps in finding them); but there is more: the numbers also have some very striking properties. Here is one, which crucially underlies the phenomenon explored in the article *Connections between Geometry and Number Theory*.

Let p be any prime number greater than 5, and let the recurring decimal corresponding to $1/p$ be computed; it will always be a pure recurring decimal. Let N be the repetend of this decimal. The number of digits in N could be odd or even. If the number of digits in N is even, say $2k$, then let A and B be the k-digit numbers obtained by 'slicing' N into two halves. *Then the sum $A + B$ is a number made up only of nines. That is,* $A + B = 10^k - 1$. Here are three examples of this remarkable phenomenon which goes by the name of *Midy's theorem*.

- If $p = 7$ then $1/p = 0.\overline{142857}$, so $N = 142857$ which has an even number of digits (with $2k = 6$). Slicing the repetend into two, we get $A = 142$ and $B = 857$. Observe that $A + B = 999 = 10^3 - 1$.
- If $p = 13$ then $1/p = 0.\overline{076923}$, so $N = 076923$ which has an even number of digits (with $2k = 6$). Slicing the repetend into two, we get $A = 076$ and $B = 923$. Observe that $A + B = 999 = 10^3 - 1$.
- If $p = 17$ then $1/p = 0.\overline{0588235294117647}$, so $N = 0588235294117647$ which has an even number of digits (with $2k = 16$). Slicing the repetend into two, we get $A = 05882352$ and $B = 94117647$. Observe that $A + B = 99999999 = 10^8 - 1$.

In a future issue of *At Right Angles* we shall explore this beautiful theorem and some of its extensions.

Problems for Solution

Problem II-2-F.1 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is 3 times the original one.

Problem II-2-F.2 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is 9 times the original one.

Problem II-2-F.3 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is $1\frac{1}{2}$ times the original one.

Solutions of Problems from Issue-II-1

Problem II-1-F.1 *Solve the cryptarithm* $\overline{EAT} + \overline{THAT} = \overline{APPLE}$.

It is immediate that $A = 1$ and $T = 9$. This yields $E = 8$ and $L = 3$. Also, $P = 0$ as the sum of a 3-digit number and a 4-digit number cannot exceed 11000. This yields $H = 2$, and now all the digits have been found: $819 + 9219 = 10038$.

Problem II-1-F.2 *Solve the cryptarithm* $\overline{EARTH} + \overline{MOON} = \overline{SYSTEM}$.

The answer for this cannot be unique because the variables H and N (the two units digits) can be swapped with no ill effects. Other than this indeterminateness, however, the solution is unique:

$$97258 + 4336 = 101594,$$
$$97256 + 4338 = 101594.$$

We leave the derivation to the reader.

Problem II-1-F.3 *Given that* $\overline{IV} \times \overline{VI} = \overline{SIX}$, *and* $\overline{SIX}$ *is not a multiple of* 10, *find the value of* $\overline{IV} + \overline{VI} + \overline{SIX}$.

Since $X \neq I, V$ it follows that $I \neq 1, V \neq 1$. Since $\overline{IV} \times \overline{VI} > 101(I \cdot V)$ and $\overline{SIX}$ is a three-digit number, it follows that $I \cdot V < 10$. Since $I > 1$, $V > 1$, $I \neq V$ we get $\{I, V\} = \{2, 3\}$ or $\{2, 4\}$. The latter does not yield a solution since $24 \times 42 > 1000$, but the former does fit: $32 \times 23 = 736$. So the code is: $I = 3$, $V = 2$, $S = 7$, $X = 6$, giving $\overline{IV} + \overline{VI} + \overline{SIX} = 32 + 23 + 736 = 791$. Note that the information that '$\overline{SIX}$ is not a multiple of 10' has turned out to be superfluous.

Problem II-1-F.4 *Explain why the numbers* 1, 121, 12321, 1234321, 123454321, *...are all perfect squares.*

It is immediate that $1 = 1^2$, $121 = 11^2$, $12321 = 111^2$, and so on. To see why the digits build up in that pattern simply examine the underlying long multiplication. For example, here is 111×111:

$$\begin{array}{r} 1\;1\;1 \\ \times\quad 1\;1\;1 \\ \hline 1\;1\;1 \\ 1\;1\;1\;\; \\ 1\;1\;1\;\;\;\; \\ \hline 1\;2\;3\;2\;1 \\ \hline \end{array}$$

Of course, the pattern will break after the number of digits exceeds 9.

Problem II-1-F.5 *Explain why the numbers* 1089, 110889, 11108889, 1111088889, *…are all perfect squares.*

We observe that $1089 = 33^2$, $110889 = 333^2$, $11108889 = 3333^2$, and so on. Let us see why this pattern persists. Let

$$A_n = \underbrace{333\ldots3}_{n\text{ threes}}.$$

Then:

$$A_n^2 = (333\ldots3)^2 = (111\ldots1) \times (999\ldots9)$$

$$= (111\ldots1) \times (10^n - 1)$$

$$= \underbrace{111\ldots1}_{n\text{ ones}}\underbrace{000\ldots0}_{n\text{ zeroes}} - \underbrace{111\ldots1}_{n\text{ ones}}$$

The subtraction clearly yields the number

$$\underbrace{111\ldots1}_{(n-1)\text{ ones}}\ 0\ \underbrace{888\ldots8}_{(n-1)\text{ eights}}\ 9,$$

which has the stated form.

A 'Least Sum' Divisibility Problem

In this short note we solve the following problem from the Regional Mathematics Olympiad (RMO) of 2006.

Given that a and b are positive integers such that $a + 13b$ is divisible by 11 and $a + 11b$ is divisible by 13, find the least possible value of $a + b$.

Attempting to solve the problem by 'brute force' does not seem satisfactory; we need a more insightful approach. We shall look for a way to generate pairs (a, b) of positive integers having the required divisibility properties, and thereby find the pair with least sum.

Since $11 \mid a + 13b$ it follows that $11 \mid a + 2b$. (Recall that '|' is the symbol for divisibility; e.g., we have $4 \mid 12$ but $5 \nmid 11$.) Similarly, since $13 \mid a + 11b$ we have $13 \mid a - 2b$. Let

$$\begin{cases} a + 2b = 11x, \\ a - 2b = 13y, \end{cases}$$

where x, y are integers. Solving this pair of simultaneous equations for a and b we get:

$$a = \frac{11x + 13y}{2}, \quad b = \frac{11x - 13y}{4},$$

and hence:

$$a + b = \frac{33x + 13y}{4}.$$

Since a and b are integers we see that x and y are either both odd or both even, and their sum must be a multiple of 4. (For: $4 \mid 33x + 13y$, hence $4 \mid x + y$.) Also, since $a > 0$ and $b > 0$ we must have

$$x > 0, \quad -\frac{11x}{13} < y < \frac{11x}{13}.$$

In any case we must have $y < x$. (Note that y can be negative.) Subject to these conditions we list in Table 1 some of the possibilities for x and y, and hence for a and b. For each value of x we have listed all possible values of y that yield integer values for a and b.

x	y	a	b	$a+b$
3	1	23	5	28
4	0	22	11	33
5	3	47	4	51
5	−1	21	17	38
6	2	46	10	56
6	−2	20	23	43
7	5	71	3	74
7	1	45	16	61
7	−3	19	29	48

The table suggests that the least possible value of $a + b$ subject to the stated conditions is **28**. We justify that this is so by observing that since $-11x < 13y < 11x$, the value of $33x + 13y$ lies between $33x - 11x$ and $33x + 11x$, i.e., between $22x$ and $44x$, and hence that

$$\frac{11x}{2} < a + b < 11x.$$

So if $x > 6$ the value of $a + b$ cannot drop below 33, and if $x = 5$ the value of $a + b$ cannot drop below $27\frac{1}{2}$ (and hence it cannot drop below 28, as it is a integer).

Since we have already achieved a value of 28 with $x = 3$ and $y = 1$, this itself must be the least possible value.

A graphical view

It is possible to view this problem in graphical terms. We consider the problem posed in the following manner: *Given that a and b are positive integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$, find the least possible value of $a + b$.*

In Figure 1 we have sketched the lines $a + 2b = 11k$ for $k = 0, \pm1, \pm2, \pm3, \ldots$ (blue, dashed), and the lines $a - 2b = 13k$ for $k = 0, \pm1, \pm2, \pm3, \pm4, \ldots$ (red, dashed). Points with non-negative integer coordinates which are part of both families of lines have been shown as heavy black dots. These correspond to the pairs (a, b) of non-negative integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$.

To find the solution with the least $a + b$ value we imagine the line $a + b = d$ drawn for increasing values of d (starting with $d = 0$), advancing across the plane; we want the least value of d for which the line passes through one of the heavy dots. It is clear that the point which this line will pass through is the one marked 'Desired point' in the graph.

From the graph we can also make out the next smallest value taken by $a + b$ (after 28). It is clearly $22 + 11 = 33$. And the one after that is $21 + 17 = 38$.

Remark. The graph reveals an important feature of the problem which the purely algebraic solution did not: the fact that the pairs (a, b) of non-negative integers which satisfy the given conditions fall on a family of lines with slope -6. Thus, we have the points $(23, 5)$, $(22, 11)$, $(21, 17)$, …which lie on the line $6a + b = 143$; the points $(47, 4)$, $(46, 10)$, $(45, 16)$, $(44, 22)$, …which lie on the line $6a + b = 286 = 2 \times 143$; and so on. (These lines have been shown in green.) Therefore we have the following interesting result which is far from obvious:

If a and b are non-negative integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$, then we have $143 \mid 6a + b$.

It is a nice exercise to prove this property algebraically, without recourse to the graph.

In closing we make the following remark: Graphs — and pictures in general — often allow us to *see* things, to spot properties of various kinds. Once seen, they may be proved rigorously using algebra. But the initial seeing (a crucial first step) is far more difficult to come by if one sticks only to algebra. Herein lie the importance and power of diagrams and well drawn pictures.

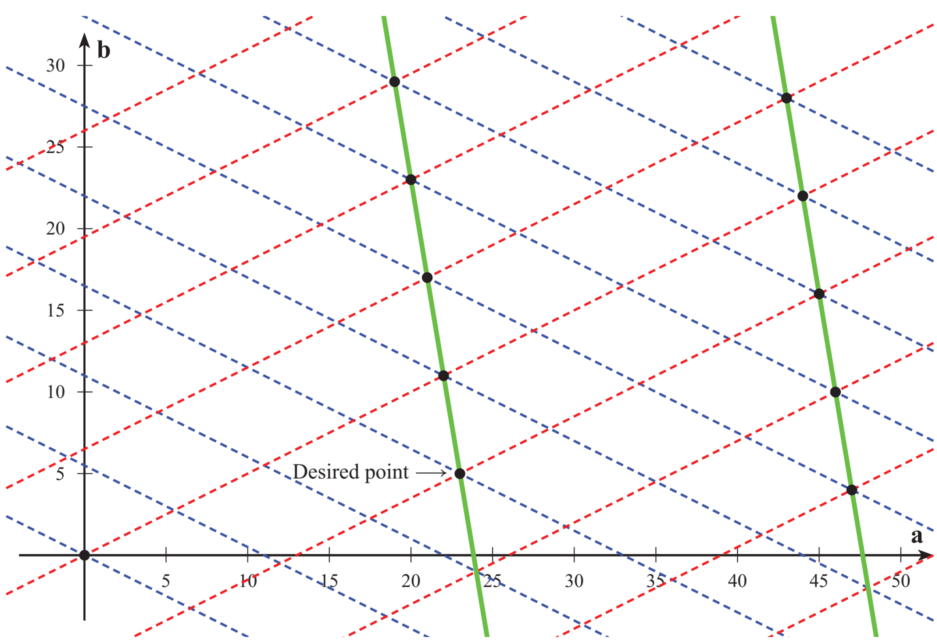

Figure 1.