Problems for the Middle School

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problem corner

Problems for Solution

Problem II-2-M.1
Find all natural numbers n such that the quantity

$$n^4 - 4n^3 + 22n^2 - 36n + 18$$

is a perfect square. (China Western Math Olympiad, 2002)

Problem II-2-M.2
A railway line is divided into 10 sections by the stations $A, B, C, D, E, F, G, H, I, J, K$. The distance from A to K is 56 km. A trip along any two successive sections never exceeds 12 km. A trip along any three successive sections is at least 17 km. What is the distance between B and G? (Swedish Math Contest, 1993)

Problem II-2-M.3
In right angled triangle ABC, with BC as hypotenuse, suppose $AB = x$ and $AC = y$ where x and y are positive integers. Squares $APQB$, $BRSC$ and $CTUA$ are drawn externally on the sides AB, BC and CA, respectively. When QR, ST and UP are joined, a convex hexagon $PQRSTU$ is formed. Let k be its area. Prove that $k \neq 2013$.

Problem II-2-M.4
The numbers $1, 2, 3, \ldots, n$ are arranged in a line in such a way that each number is either strictly bigger than all the numbers to its left, or strictly smaller than all the numbers to its left. In how many ways can this be done? (21-st Canadian Math Olympiad,1989)

Problem II-2-M.5
If a, b, c are real numbers such that $1/a + 1/b + 1/c = 1/(a + b + c)$, show that the following is true for any positive integer n:

$$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{a^{2n+1} + b^{2n+1} + c^{2n+1}}.$$

Solutions of Problems in Issue-II-1

Solution to problem II-1-M.1 *Two distinct two-digit numbers a and b are chosen (a > b). Their GCD and LCM are two-digit numbers, and a/b is not an integer. What could be the value of a/b?*

Let $c = \text{GCD}\,(a, b)$ and $d = \text{LCM}\,(a, b)$; let $a = a'c$ and $b = b'c$. Then: (i) $a' > b'$; (ii) a', b' are coprime; (iii) $d = a'b'c$; (iv) c, $a'c$, $b'c$, $a'b'c$ lie between 10 and 99; (v) a'/b' is not an integer. Since $c \geq 10$ and $a'b'c \leq 99$ we also have: (vi) $a'b' < 10$. So a', b' are digits.

Applying (i), (ii), (v), (vi) we find that just one pair is left: $(a', b') = (3, 2)$. It follows that $a/b = 3/2$.

We can say more. We have: $a = 3c$, $b = 2c$, $d = 6c$. Since $c, 2c, 3c, 6c$ are two-digit numbers, it follows that $10 \leq c \leq 16$. Hence the possibilities for (a, b) are the following: $(30, 20)$, $(33, 22)$, $(36, 24)$, $(39, 26)$, $(42, 28)$, $(45, 30)$ and $(48, 32)$.

Solution to problem II-1-M.2 *The sum of a list of* 123 *positive integers is* 2013*. Given that the LCM of those integers is* 31*, find all possible values of the product of those* 123 *integers.*

As the LCM of the numbers is 31, each number is a divisor of 31. As 31 is prime, its only divisors are 1 and 31. Hence each number in the list is 1 or 31. Let the number of 1s in the list be x, and the number of 31s be y. Then $x + y = 123$ and $x + 31y = 2013$. Solving these equations for x and y we get $x = 60$ and $y = 63$. So the list is:

$$\underbrace{1, 1, 1, \ldots, 1, 1}_{\text{60 of these}},\ \underbrace{31, 31, 31, \ldots, 31, 31}_{\text{63 of these}}.$$

Solution to problem II-1-M.3 *Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $a + b = c + d$, show that: (i) a is a divisor of b; (ii) $a^3 + b^3 = c^3 + d^3$.*

Let $a = ca'$ and $b = cb'$; then a', b' are coprime, and $a' \leq b'$. As the product of two numbers also equals the product of their GCD and LCM, we have $cd = a'cb'c$, i.e., $d = a'b'c$. Since $a + b = c + d$ it follows that $ca' + cb' = c + ca'b'$, i.e., $a' + b' = 1 + a'b'$. This leads to:

$$a'b' - a' - b' + 1 = 0, \qquad \therefore (a' - 1)(b' - 1) = 0,$$

hence at least one of a', b' equals 1. Since $a' \leq b'$, it follows that $a' = 1$. Hence $a = c$, implying that a is a divisor of b, and $d = b$. Both (i) and (ii) now follow.

Solution to problem II-1-M.4 *Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $ab = c + d$, find all possible values of a and b.*

Since the product of two numbers also equals the product of their GCD and LCM we have $ab = cd$, hence $cd = c + d$. This may be written as $cd - c - d + 1 = 1$, giving $(c - 1)(d - 1) = 1$. Hence $c - 1 = 1 = d - 1$, i.e., $c = 2 = d$. As the GCD and LCM are both equal to 2, the numbers must be 2, 2. That is, $a = 2 = b$.

Solution to problem II-1-M.5 *Let a and b be two positive integers, with $a \leq b$, and let their GCD be c. Given that $abc = 2012$, find all possible values of a and b.*

Let $a = ca'$ and $b = cb'$. Then a', b' are coprime. We are told that $abc = 2012$. Hence $a'b'c^3 = 2012$. Now the prime factorization of 2012 is $2012 = 2 \times 2 \times 503$. So we have $a'b'c^3 = 2 \times 2 \times 503$, with GCD $(a'b') = 1$. Since 2012 is not divisible by a cube larger than 1, it follows that $c = 1$, i.e., a, b are coprime. Since $a \leq b$ (given), the possibilities for (a, b) are $(1, 2012)$ and $(4, 503)$.

Solution to problem II-1-M.6 *Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $d - c = 2013$, find all possible values of a and b.*

Let $a = ca'$ and $b = cb'$; then $d = a'b'c$, so the information given yields: $a'b'c - c = 2013$, i.e., $c(a'b' - 1) = 2013$. Hence c is a divisor of 2013. Now the prime factorization of 2013 is $3 \times 11 \times 61$. Hence the divisors of 2013 are the following: 1, 3, 11, 33, 61, 183, 671, 2013. (There are 8 divisors.) The possibilities are thus:

c	1	3	11	33	61	183	671	2013
$a'b' - 1$	2013	671	183	61	33	11	3	1
$a'b'$	2014	672	184	62	34	12	4	2

Each value of $a'b'$ in the last line leads to possible values of (a, b). If $a'b' = 2$ then $(a', b') = (1, 2)$, so $(a, b) = (2013, 4016)$. If $a'b' = 4$ then $(a', b') = (1, 4)$, so $(a, b) = (671, 2684)$. If $a'b' = 12$ then $(a', b') = (1, 12)$ or $(3, 4)$, so $(a, b) = (183, 2196)$ or $(549, 732)$. And so on — all the possibilities can be thus listed, one by one.