Viewpoints

Angles and Demons

Presenting two rarely discussed facets of angles which straddle two different areas. The first part goes straight into the difficulties of measurement, and the second part discusses alternative ways of measuring angles. Read this to bring more to your class than just the historical need to measure angles and its applications in real life.

The Editors

The difficulties children encounter (during a formal study of angles) might imply that angle and turn measure need not be introduced to young children. However, there are valid reasons to include these as goals for early childhood mathematics education. First, children can and do compare angle and turn measures informally. Second, use of angle size, at least implicitly, is necessary to work with shapes; for example, children who distinguish a square from a non-square rhombus are recognizing angle size relationships at least at an intuitive level. Third, angle measure plays a pivotal role in geometry throughout school and laying the groundwork early is a sound curricular goal. Fourth, the research indicates that although only a small percentage of students learn angles well through elementary school, young children can learn these concepts successfully. Source: [1]

Reading further in [1], one sees a learning trajectory for angle measurement which starts with an intuitive angle builder (2–3 years), an implicit angle user (4–5 years), an angle matcher (6 years), an angle size comparer (7 years), and an angle measurer (8+ years).

This article focuses on angle measurement, which according to the above trajectory should be taught in the third standard but which continues to give students difficulties two or even three years later

To most grown-ups, angles present no difficulty. An angle has a vertex and two arms spread apart to a certain degree called the 'measure' of the angle; a handy instrument called a protractor can be used to measure that degree. This definition is present in many textbooks. It seems such a simple concept that one cannot imagine anyone having difficulty with it. Workbooks dedicate a page or two to an introduction on angles and hurry on to problems on drawing and measurement and naming parts of an angle. But ask the students to measure the angle of an inverted cone and most of them struggle with orienting their protractors correctly. Or show two equal angles with different arm lengths and ask them to say which is greater; a good many of them will pick the one with the bigger arms. Or study the situation depicted in Figure 1, where the student thinks that the baseline has to be completely covered by the protractor.

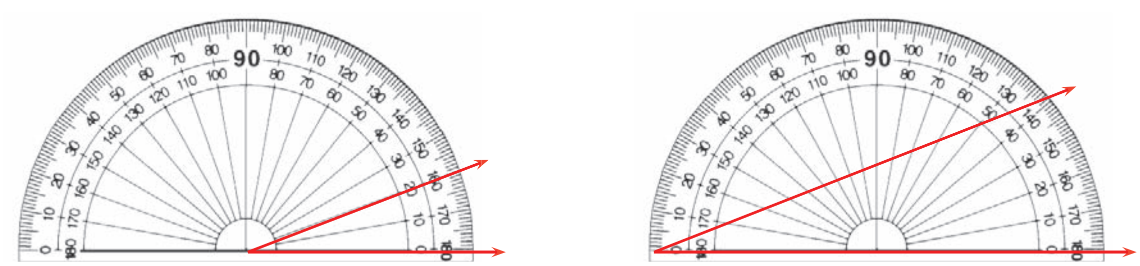
Why do such misconceptions arise? Is it because, from the beginning, we inundate children's minds with words like vertex, line segment, ray and so on and neglect practical tasks associated with measurement? Pick up a protractor and examine it. Is it really easy to use with its mass of lines and markings (clockwise and counter-clockwise) and numbers? In fact, it is so complicated that it is a miracle that children learn to use it at all!

In this article we present a sequence of ideas that introduces young learners to angles. It is guided by the belief that anything which relates to the tangible world of children is going to have better learning outcomes than otherwise.

Playing with angles. Angles have been defined from two perspectives — as a 'shape' formed by two rays extending from one point, or as a 'rotation' or 'turn'. Students sometimes think of these as different concepts. Activities dealing with angles should encompass both the notions, so that students appreciate the intrinsic meaning of the term.

Using a circle of paper folded into quarters (and therefore with rounded edges), the teacher can demonstrate how to make a right angle. By aligning it against different angles, students can grasp the right way to compare two angles (Figure 2). The device also serves as a rudimentary protractor. The same shape can be folded or unfolded to form smaller and bigger angles. After this, introducing the terms 'acute' and 'obtuse' is a matter of association.

An interesting use of this folded shape is to illustrate invariance of angle with arm length, a concept even middle-school students sometimes



A 20° angle . . .   . . . or a 40° angle?

Figure 1. 20° or 40°? Some students would say it is the latter.

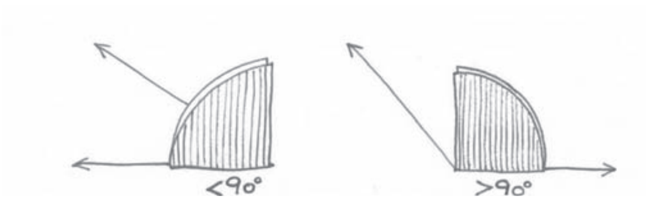


Figure 2.

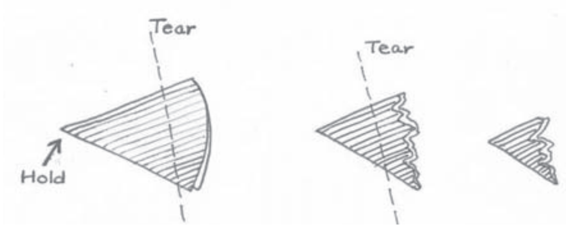


Figure 3.

grapple with. Hold it at the vertex and tear the paper (Figure 3). QED!

Another way is to use a string and two straws (Figure 4). Not only can the invariance of angle with arm length be demonstrated by moving the straws along the arms, but one can also demonstrate the idea that an angle can be formed 'in the imagination', with the vertex not visible. This is a difficulty that students in grades 9 and 10 experience while studying the topic of 'Heights and Distances' in trigonometry.

Figure 4.

Do you have a class with any kinesthetic learners? Then introduce the notion of angle with the help of a game: 'Angle Yoga'. Starting from the zero position and keeping one arm fixed, call out 'Right', 'Acute' or 'Obtuse' and move the other arm accordingly. As they do this, children realize several things, e.g., children with different arm lengths can show the same angle; there can be a range of correct angles for acute and obtuse angles; the fixed arm need not be horizontal or vertical; angles can be oriented differently. Most importantly, they learn the art of estimation using their arms to form an angle.

An interesting way of measuring rotation is to use the classroom door (Figure 5). The teacher marks angles on the floor from 0° to 90°, at intervals of 15° or 30°. This becomes a self-learning tool; children interact with it and learn. They may not immediately understand what the degree symbol is or why some markings are missing. Some may wonder what happens if the door opens even further: how does one go about measuring the angle then?

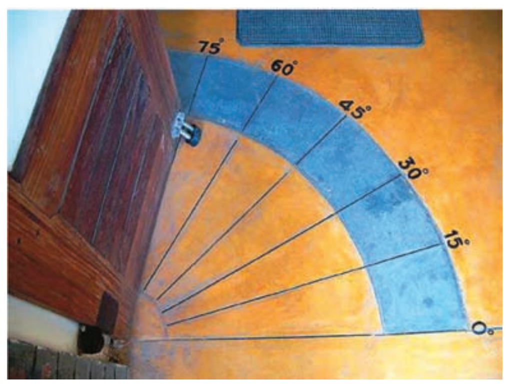


Figure 5. Source: http://business.outlookindia.com/printarticle.aspx?267253

(Which makes me wonder: Why are protractors not made with a slim metal strip that swivels from the centre and opens up from 0° to 180°?)

At this point, terminology such as *ray*, *vertex* and *line segment* can be introduced. Since students are familiar with the measurement of length using the iteration of a unit, the degree as a unit of measurement of angles should be acceptable to them. Geogebra is an excellent tool for students to understand the concept of rotation.

Different ways of measuring angles.

Now that students have learnt to measure angles using the protractor, they can investigate other ways of measuring angles and the advantages and disadvantages of these methods.

Maybe ancient geometers measured angles by fitting a line segment between the arms at a standard distance? Let's see what this leads to.

Suppose that, in order to measure $\measuredangle AOB$, points C and D are marked, one on each arm, 1 unit length from the vertex, and segment CD is drawn. Then the length of CD is taken to be a measure of $\measuredangle AOB$ (Figure 6). We call this the **chord method** to measure angles.

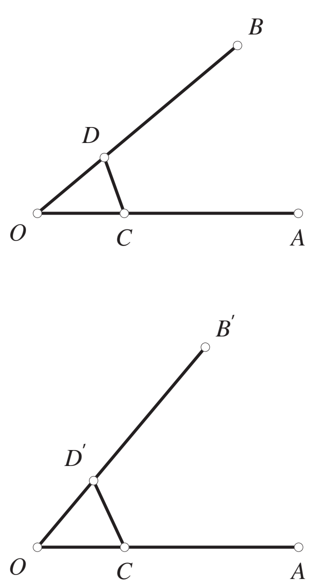


Figure 6. Here OC = OD = OD′.
If CD′ > CD then $\measuredangle$AOB′ > $\measuredangle$AOB, and conversely

It can be checked that this approach does preserve the *order relation*. In other words, if $\measuredangle AOB < \measuredangle AOB'$ then $CD < CD'$; and conversely. To see why, we apply the 'inequality form of the SAS congruence theorem' which states the following (we only consider the form applicable to isosceles triangles, as that is all we need): Let $\measuredangle ABC$ and $\measuredangle PQR$ *be isosceles, with* $AB = AC = PQ = PR$. *Then: if* $\measuredangle A < \measuredangle P$, *then* $BC < QR$; *and if* $BC < QR$, *then* $\measuredangle A < \measuredangle P$. This can be proved using pure geometry, but we leave the proof to you. (Some readers may prefer the following trigonometric proof. In an isosceles ΔABC in which $b = c$, we have $a = 2b \sin A/2$. Since b is fixed and $\sin x$ is an increasing function of x over the interval from 0° to 90°, it follows that a increases when $\measuredangle A$ increases from 0^0 to 180^0; and conversely. The same conclusion is reached if we use the cosine rule which yields: $a^2 = 2b^2(1\text{-}\cos A)$, but now we use the fact that $\cos x$ is a decreasing function of x over the interval from 0° to 180°.)

So the chord method of measuring angles preserves the order relation. But it fails to pass a second test which is as important: *additivity*. To see what this means, consider a pair of adjacent angles, $\measuredangle AOB$ and $\measuredangle BOC$, which share an arm OB (Figure 7). Since $\measuredangle AOC$ is the union of $\measuredangle AOB$ and $\measuredangle BOC$, and there is no 'overlap' between the latter two angles, it is reasonable to demand that the measure of $\measuredangle AOC$ should be equal to the sum of the measures of $\measuredangle AOB$ and $\measuredangle BOC$. But does this requirement hold good for the chord measure?

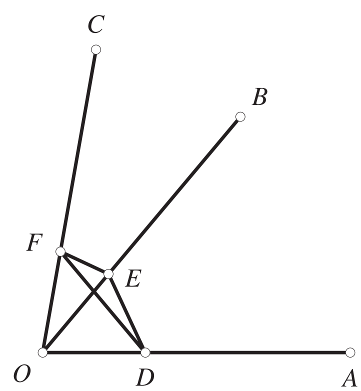


Figure 7.

Let D, E, F be points on the rays OA, OB, OC such that $OD = OE = OF = 1$ unit. By definition, the chord measures of $\measuredangle AOB$, $\measuredangle BOC$ and $\measuredangle AOC$ are the lengths DE, EF and DF, respectively. Is it true that $DE + EF = DF$? Clearly not. In fact we will always have $DE + EF > DF$, for two sides of a triangle are together always greater than the third side (here applied to ΔDEF). So the sum of the measures of $\measuredangle AOB$ and $\measuredangle BOC$ is *greater* than the measure of $\measuredangle AOC$. We see from this line of reasoning that the chord measure of an angle fails the test of additivity.

(Note: The above argument assumes that points *D, E, F* in Figure 7 do not lie in a straight line. But how can we be sure that they do not lie in a line? If we do not provide a justification for this, then what we have said is incomplete. Readers are asked to find a proof on their own.)

We do not know whether the ancients considered chord length as a candidate for angle measure. The measure they did adopt is the one we use today, and it possesses both the desired attributes —the order relation and the additivity property. It is based on **arc length.** Here, given an $\measuredangle AOB$, we mark points *C* and *D*, one on each arm, at 1 unit length from the vertex, and draw the circle with centre *O* and passing through *C* and *D*. Then *the length of arc CD is taken to be a measure of* $\measuredangle AOB$ (Figure 8).

Let us see how additivity is handled by this definition. In Figure 9 we see $\measuredangle AOB$ and $\measuredangle BOC$ which share an arm *OB*. As in Figure 7 the two angles have no overlap. Their arc measures are the lengths of arcs *DE* and *EF* which are both part of the circle with radius 1 unit, centred at *O*. The arc measure of $\measuredangle AOC$ is the length of the single arc *DF*. Is the length of arc *DF* equal to the sum of the lengths of arcs *DE* and *EF?* Clearly yes, as the arcs are all part of a single circle, and arc *DF* is simply the union of the two smaller non-overlapping arcs.

Arc measure of an angle is less natural than chord measure, but one begins to appreciate its elegance and advantages as one studies it more deeply.

In conclusion we may say that constructing, interpreting and recognizing shortcomings or gaps in definitions are all teaching and learning opportunities where teacher and learner can work together for greater understanding. It is when we see the inadequacies of attempted definitions that we begin to recognize the beauty and economy of existing definitions.

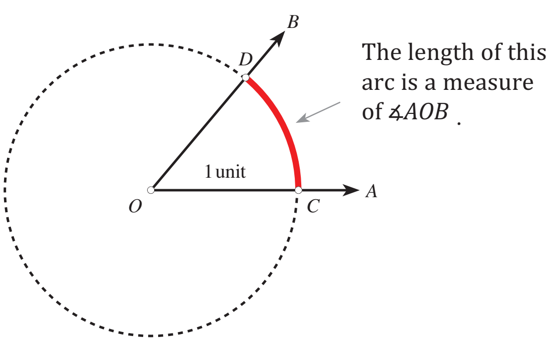


Figure 8.

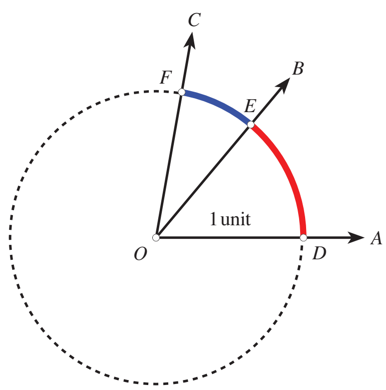


Figure 9.

Acknowledgements.

This article is the outcome of several absorbing discussions on different platforms. Anupama, a math resource person at the University Resource Centre of Azim Premji University presented a paper on ‘Angles’ at a seminar, following which there was an animated discussion in the online math learning group, focusing on misconceptions that students may have regarding angles and how a teacher can address them. The math learning group gratefully acknowledges the contributions of Dr. Ravi Subramaniam (HBCSE), Dr. Shailesh Shirali (CoMaC), Dr. Hridaykant Dewan (VBS), Ramchandar Krishnamurthy (APF), Rajveer Sangha and Jyothi Thyagarajan to this discussion.

References

[1] *From Learning and Teaching Early Math —The Learning Trajectories Approach Studies in Mathematical Thinking, by Clements and Sarama*

number crossword-3

When the students of DRIK (Dwaraknath Reddy Institutes for Knowlededge) Patashala, Chittor came across the number crosswords in At Right Angles, (Issue I-1 and Issue I-2), they got down to solving them with great enthusiasm. But they ran into unexpected difficulties – not with the mathematics but with the language used. Though they studied in an English medium school, understanding the language in which the clues were framed seemed a challenge! However, when with a little help they managed to solve the crossword, there was no doubting the fact that number crosswords had found many enthusiastic converts. Their teachers too realised that here was an interesting way to improve language as well as mathematical skills.

When this incident was recounted to the creator of the crosswords, Mr D.D. Karopady, he came up with the interesting suggestion of having the next crossword created by these very students, which we now present to you for solution!

■	1		■	2	3	■
4		■	5	■	6	7
	■	■		■	■	
■	8		■	9		■
10	■	■	11	■	■	12
13	14	■		■	15	
■	16		■	17		■

CLUES ACROSS :

1. 'L' IN ROMAN NUMERALS
2. 1221 TIMES ------ IS EQUAL TO 111111
4. 7D PLUS 5
6. 15D+7D+12D-25
8. LAST TWO DIGITS OF THE REVERSED MEASURE OF A LINEAR ANGLE
9. THE PRODUCT OF 15D AND THE FIRST DIGIT OF 7D
13. THE LCM OF 12 AND 30
15. 12D MINUS 30
16. 3D TIMES 3
17. 15D PLUS 5^2 MINUS 5

CLUES DOWN :

1. 4A MINUS 10
3. CUBE ROOT OF 2197 PLUS 6
4. PRIME NUMBER MADE WITH 1 AND 6
5. LAST TWO DIGITS OF THE MEASURE OF A GROSS
7. 13A PLUS 2
10. 4 SQUARED
11. SUM OF DIGITS OF THE RAMANUJAN NUMBER
12. 4 CUBED MINUS CUBE ROOT OF 4096
14. LOWEST AND HIGHEST NUMBERS AMONG 2,5,0,3,4,1
15. FIFTH PRIME NUMBER

DRIK Patashala in Chittoor, Andhra Pradesh was established in 2006 with the vision to ensure that children from urban slums and villages accessed their rights to education. The school now has 84 children chosen from the most neglected circumstances. Experiential learning, exposure trips, lots of music, dance, art and games are all part of an evolving and empowering curriculum which takes education beyond schooling for these children.