How to discover 22/7 and other rational approximations to π

in the classroom

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Introduction

In math and physics books, we often find the words "Take $\pi = 22/7$". You and I, dear reader, know that this cannot be right. We know perfectly well that π is an irrational number, and so cannot *equal* 22/7; for 22/7 is clearly a ratio o integers and therefore a rational number. So the best we can say is $\pi \approx 22/7$, that is, π is *approximately* 22/7.

In this article, I will explain a method to find such rational approximations for π and other irrational numbers. The key idea here is to use a calculator (or a spreadsheet program) to find what is called a *continued fraction* for an irrational number.

Keywords: *Pi, approximation, rational, continued fraction, square root, spreadsheet, Excel*

The continued fraction for π

It is easy to see that:

$$\pi = 3.14159265\ldots = 3 + 0.14159265\ldots$$
$$= 3 + \frac{1}{1/0.14159265\ldots}$$

We now compute $1/0.14159\ldots$ in the denominator, using a calculator, and obtain:

$$\pi = 3 + \frac{1}{7.06251331\ldots}$$

Repeating these steps, we obtain:

$$\pi = 3 + \frac{1}{7 + 0.06251331\ldots} = 3 + \cfrac{1}{7 + \cfrac{1}{1/0.06251331\ldots}}$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15.99659440\ldots}} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1/0.99659440\ldots}}}$$

$$= \cdots$$

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}.$$

This process can be continued indefinitely, and we obtain what is called a *continued fraction* representation of π. To get approximations of π, chop off the continued fraction suitably to get:

$$\pi \approx 3 + \frac{1}{7} = \frac{22}{7};$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106};$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = \frac{355}{113}.$$

The approximations $22/7$ and $355/113$ are quite popular, and have been known for thousands of years.

So now you know how to discover $22/7$ and other rational approximations to π. Let us try the same thing with another familiar irrational number, namely $\sqrt{2}$.

The square root of 2

Unlike π, the continued fraction of $\sqrt{2}$ has a beautiful pattern. Here are the calculations:

$$\begin{aligned}\sqrt{2} &= 1.414213562\ldots = 1 + 0.414213562\ldots\\ &= 1 + \frac{1}{1/0.414213562\ldots} = 1 + \frac{1}{2.414213562\ldots}\\ &= 1 + \cfrac{1}{2 + \cfrac{1}{1/.414213562\ldots}} = 1 + \cfrac{1}{2 + \cfrac{1}{2.414213562\ldots}}\\ &= \cdots\end{aligned}$$

Notice that the $2.414\ldots$ has occurred earlier. So you would expect that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

This is an infinite continued fraction representation of $\sqrt{2}$, and is surely much nicer than its decimal expansion $1.414213562\ldots$.

One can *prove* that the pattern repeats, and also avoid the use of a calculator, by noticing the following equality which happens to be exact:

$$\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}.$$

Replace the $\sqrt{2}$ on the RHS by the expression

$$1 + \frac{1}{1 + \sqrt{2}}$$

and see if you can tell why the pattern repeats!

I leave it to you to chop off the terms of the continued fraction and find nice rational approximations for $\sqrt{2}$. The first few approximations are: $3/2$, $7/5$, $17/12$, $41/29$ and $99/70$.

You may find it interesting to find the continued fractions for $\sqrt{3}$ and $\sqrt{5}$ in the same way. The patterns are every bit as nice as those in the continued fraction for $\sqrt{2}$. In fact, you are well equipped to find out rational approximations of a host o irrational numbers, such as e, π^2, e^{π}. Why don't you try some experiments of your own?

Some experiments on the simplest infinite continued fraction

You should also try your hand at the simplest of all continued fractions:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}.$$

Chop off the continued fractions and calculate a few fractions. The fractions are closely related to the famous *Fibonacci sequence* $1, 1, 2, 3, 5, 8, 13, \ldots$.

Can you figure out the number to which these approximations converge? Here's a hint: It is a number that stars prominently in the movie *The da Vinci Code*. Another hint: It is the only positive number that is one more than its reciprocal.

Notes

- Of course, the simplest way to approximate π by a rational number would be to take the first few digits of its decimal expansion. For example, 3.14 is a perfectly legal rational approximation of π. And 3.14159 is an even better approximation. But surely, $22/7$ is much prettier!

- It is said that Archimedes found the approximation $22/7$ of π by laboriously approximating a circle with a 96-sided polygon. Google the phrase "approximations to pi" to find the story of approximations of π from ancient times to Ramanujan; and to a record computation of trillions of digits of π.

- One can prove that the continued fractions of numbers such as $\sqrt{2}, \sqrt{3}$ etc. are nice, in the sense that the numbers repeat periodically, in much the same way as decimal expansions of some rational numbers.

- We have not considered whether an infinite object such as

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}$$

  has a meaning. After all, in the finite amount of time accorded to us in this lifetime, we cannot hope to perform all the calculations that are indicated by the '…'.

  The key idea is to consider a sequence of fractions x_n that are obtained by chopping off the continued fraction, and to show this sequence *converges*.

  One way to convince yourself of this is to use a computer and calculate a number of fractions by suitably chopping the infinite continued fractions. Now find their decimal expansions, and see if they seem to be stabilizing. That is to say, if you take the difference $x_n - x_{n-1}$ of successive fractions, does the difference come close to 0 as n becomes large? This is not enough to prove the sequence converges, but it does give evidence of convergence.

  If you study the values of $x_n - \sqrt{2}$ for increasing n, you may discover another interesting feature of these approximations. But I won't tell you what it is. Find out for yourself!

- You can use this approach to approximate complicated rational numbers too. For example,

$$\frac{985}{304} \approx 3 + \frac{1}{4} = \frac{13}{4}.$$

- There are much better-looking continued fractions for π than the one given in this article. One of them, found by Lambert in 1770, is

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cdots}}}}.$$

- I could have *proved* that $\sqrt{2}$ is irrational, but then I would have been obliged to prove that π is irrational too, which is quite tough. The best such proofs involve continued fractions. Had $\sqrt{2}$ been a rational number, its continued fraction would have been finite. This is because when the process of finding continued fractions is applied on a rational number of the form p/q, it will stop after a finite number of steps. If you don't believe me, try the process on a few fractions, and discover just why it stops after a few steps.

  A nice book which contains such a proof is *Mathematics: A very short introduction* by T. Gowers, Oxford University Press (2002).

- If you liked what you saw here, then you will definitely enjoy the classic: *Continued Fractions* by C. D. Olds, Random House (1963).

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SOLUTION TO THE 'RIVER PUZZLE' GIVEN BY SAM LOYD HIMSELF

See ***page* 30** for a statement of the problem.

At their first meeting, 720 yards from one shore, the boats together have travelled a distance equal to the width of the river. When each reaches its destination, the combined distance is twice the width of the river. At their second meeting, 400 yards from the other shore, the combined distance is three times the river's width. So at the second meeting each boat has gone three times as far as it had at the first meeting.

At the first meeting one boat had travelled 720 yards, so at the second meeting it has gone three times this distance, or 2,160 yards. At this point it has covered 400 yards since doubling back, so the river is 2,160 - 400 = 1,760 yards, or one mile, wide.

The amount of time each ship consumed at the landing does not affect the problem.

Comment from the Editor. This beautiful solution tells us that Sam Loyd is called the Master for very good reason!