Exploring Fibonacci Numbers using a spreadsheet

In this article we are going to explore a very interesting sequence of numbers known as the Fibonacci sequence. The exploration of the sequence can lead to an absorbing classroom activity for students at the middle school and secondary school level. Students can explore many patterns within the sequence using a spreadsheet like MS Excel and the observations can lead to an enriching discussion in the classroom.

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The Fibonacci Puzzle

The Fibonacci puzzle was posed by Leonardo of Pisa, also known as Fibonacci (hence the name of the sequence). The puzzle describes the growth of an idealized rabbit population. A newly born pair of rabbits comprising a male and a female rabbit is put in the field and are able to mate at the age of one month. At the end of the second month the female rabbit produces a pair of rabbits (again a male and a female). Rabbits never die and every mating pair always produces a new pair every month from the second month on. How many pairs will there be in one year?

Let us try to find out the number of pairs of rabbits at the end of every month starting from the first month. To begin with there is one pair of rabbits. At the end of the first month, they mate, but there is still only 1 pair. At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the

field. At the end of the third month, the female of the original pair produces a second pair, making 3 pairs altogether in the field. Remember that the second pair which was born at the end of the second month is only able to mate at the end of the third month. At the end of the fourth month, the female of the original pair has produced yet another new pair and the female born at the end of the second month produces her first pair, making 5 pairs in all. This process continues. To obtain the number of pairs at the end of any given month, say n, we need to add the number of pairs at the end of month $n-1$ and the number of pairs at the end of month $n-2$.

Hence the Fibonacci sequence can be written in the form of the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

The first two terms of the sequence are 1 and 1. So if $F_1 = F_2 = 1$, the recurrence relation can also be written as

$$F_{n+2} = F_{n+1} + F_n.$$

Sometimes F_0 is taken to be 0. Here are the first fifteen Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.

Can you guess how large the 50th Fibonacci number will be? One can continue the process of adding pairs of consecutive terms to find the next term, but after a while it can get quite cumbersome. Let us take the help of a spreadsheet to generate the Fibonacci sequence and find its 50th term.

Generating the Fibonacci sequence on Excel: The following steps will help you obtain the sequence on Excel.

Step 1: The first step is to create a column of integers from 1 to 50 (in column A). This may be achieved in the following manner.

▪ Click on cell A2 and enter 1. Then enter = **A2 + 1** in cell A3 and drag cell A3 till A51. This will create a column of numbers 1 to 50.

Step 2: Enter 1 in cells B2 and B3. In cell B4, enter = **B3 + B2** and press enter. A double click on the corner of cell B4 will generate the Fibonacci sequence as shown in Figure 1. As you scroll down the sequence in column B, you will notice that the column width is too small to accommodate the numbers. For example the 40th Fibonacci number appears as 1.02E + 08. This means that Excel has approximated the number and the number is close to 1.02×10^8. To get the actual terms of the sequence beyond the 40th term, the column width needs to be increased. This can be done by taking the cursor to the end of the column (were the columns are named as A, B, C etc) and dragging it to the required width.

Note that the 50th Fibonacci number is 12586269025. Clearly, this is a very fast growing sequence. See Figure 2.

	A	B	C
	A1		f_x
1		Fibonacci Seq	
2	1	1	
3	2	1	
4	3	2	
5	4	3	
6	5	5	
7	6	8	
8	7	13	
9	8	21	
10	9	34	
11	10	55	
12	11	89	
13	12	144	
14	13	233	
15	14	377	
16	15	610	
17	16	987	
18	17	1597	

Figure 1: Generating the Fibonacci sequence on Excel.

40	39	63245986
41	40	102334155
42	41	165580141
43	42	267914296
44	43	433494437
45	44	701408733
46	45	1134903170
47	46	1836311903
48	47	2971215073
49	48	4807526976
50	49	7778742049
51	50	12586269025

Figure 2: A part of the Excel sheet showing the 39th to 50th terms of the Fibonacci sequence.
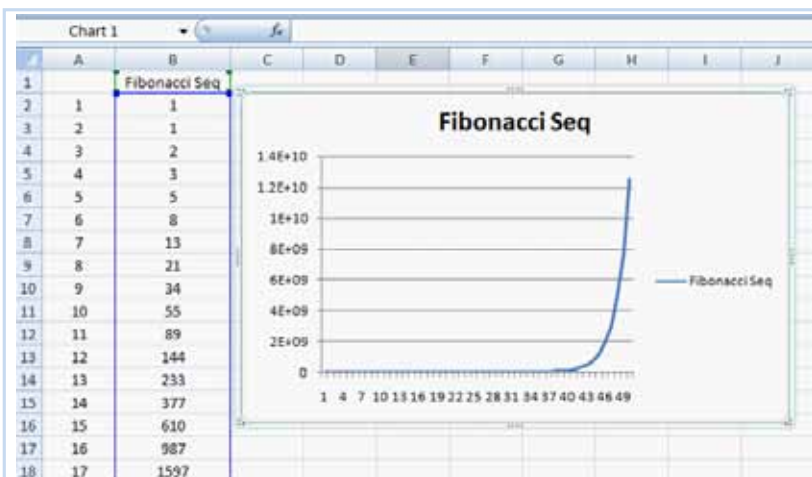
Figure 3: Graphing the Fibonacci sequence on Excel.

Step 3: We can also draw the graph of the sequence. For this, select column B by clicking on B, go to **Insert** on the toolbar, select **chart** and then **line**. See Figure 3.

As you already know, the first 15 Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610. Let us see what happens when we take the ratios of consecutive terms $\frac{F_{n+1}}{F_n}$ of the Fibonacci sequence. They are:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233}, \frac{610}{377}, \ldots ,$$

Let us compute the ratios on Excel as follows. In cell C3 of column C, enter = B3 / B2 and double click on the corner of the cell. You will observe that after a certain number of terms the ratios become steady at 1.618034.

A natural question now arises whether this value (that is, 1.618034) will remain the same if we change the initial values of the Fibonacci sequence (which are 1 and 1). Let us try to investigate by taking different starting values. Change the values in cells B2 and B3 in the Excel sheet to 4 and 7. Observe that the ratios of successive terms still approach 1.618034 (shown on Figure 5). You may investigate by taking different starting values. You will observe that the ratios of successive terms still approach 1.618034. A graph of the ratios of successive terms (obtained by selecting column C) reveals this behavior of the sequence. See Figures 5 and 6. This is indeed an interesting observation but we need to find a mathematical explanation for it.

	A	B	C	D
1		Fibonacci Seq		
2	1	1		
3	2	1	1	
4	3	2	2	
5	4	3	1.5	
6	5	5	1.666667	
7	6	8	1.6	
8	7	13	1.625	
9	8	21	1.615385	
10	9	34	1.619048	
11	10	55	1.617647	
12	11	89	1.618182	
13	12	144	1.617978	
14	13	233	1.618056	
15	14	377	1.618026	
16	15	610	1.618037	
17	16	987	1.618033	
18	17	1597	1.618034	

Figure 4: The ratios of consecutive terms of the Fibonacci sequence become steady at 1.618034 (in column C)

	A	B	C	D
1		Fibonacci Seq		
2	1	4		
3	2	7	1.75	
4	3	11	1.571429	
5	4	18	1.636364	
6	5	29	1.611111	
7	6	47	1.62069	
8	7	76	1.617021	
9	8	123	1.618421	
10	9	199	1.617886	
11	10	322	1.61809	
12	11	521	1.618012	
13	12	843	1.618042	
14	13	1364	1.618031	
15	14	2207	1.618035	
16	15	3571	1.618034	
17	16	5778	1.618034	
18	17	9349	1.618034	

Figure 5: The terms of the sequence with initial values 4 and 7.

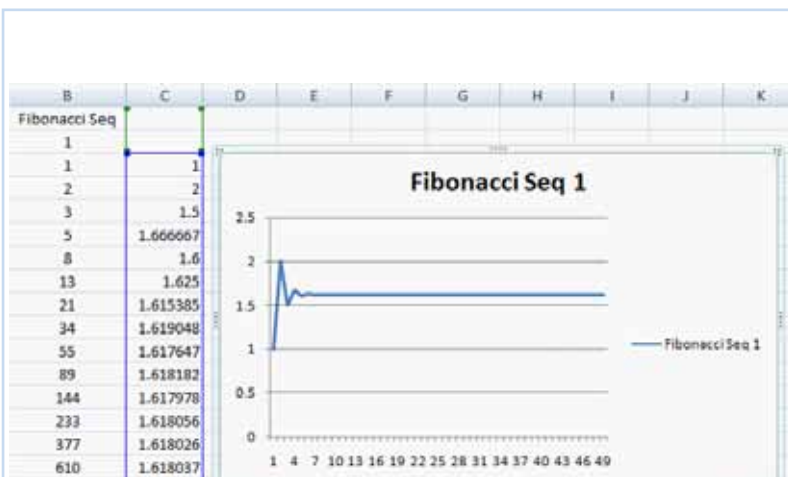| B	C
Fibonacci Seq	
1	
1	1
2	2
3	1.5
5	1.666667
8	1.6
13	1.625
21	1.615385
34	1.619048
55	1.617647
89	1.618182
144	1.617978
233	1.618056
377	1.618026
610	1.618037

Figure 6: A graph of the ratios of consecutive terms of the Fibonacci sequence becomes steady at 1.618034.

We already know that the Fibonacci sequence is given by the recurrence relation,

$$F_{n+1} = F_n + F_{n-1} \tag{1}$$

Also from the sequence of ratios we observe that after some value of n, we have:

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{2}$$

Let this ratio be equal to x. We would like to show that $x = 1.618$ up to three places of decimals.

Using (1) and (2) we get

$$\frac{F_n + F_{n-1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{3}$$

$$or \quad 1 + \frac{F_{n-1}}{F_n} = \frac{F_n}{F_{n-1}} \tag{4}$$

This implies that $1 + \frac{1}{x} = x$. Simplifying this we get the quadratic equation $x^2 - x - 1 = 0$ whose roots are

$$\frac{1 \pm \sqrt{5}}{2}.$$

We ignore the value $\frac{1 - \sqrt{5}}{2}$ as x cannot be negative.

Thus $x = \frac{1 + \sqrt{5}}{2}$ (which is approximately equal to 1.618 up to three decimal places).

This is also known as the *Golden Ratio*.

The Fibonacci sequence $F_{n+2} = F_{n+1} + F_n$ with $F_1 = F_2 = 1$ is a recursive sequence. We have also seen from our Excel exploration that the terms of the sequence grow large very quickly. But wouldn't it be nice if we had a formula for the nth Fibonacci number? After all, if we wanted the 100th or the 1000th Fibonacci number, it would be far more convenient to have a formula in which we could insert the value of n (for the 100th term, $n = 100$) and obtain the required number.

The general term of the Fibonacci sequence: To find the required formula, we need to get acquainted with a simple but useful tool. Given a sequence of numbers $x_1, x_2, x_3,...$we define the *left shift operator L as follows:*

$$L(x_n) = x_{n+1},$$

where x_n is the nth element of the sequence $x_1, x_2, x_3, ...$ If L is made to operate on this sequence, we will obtain:

$$L(x_1), L(x_2), L(x_3),....$$

This is just: $x_2, x_3, x_4,....$ i.e., the original sequence shifted to the left by one term (x_1 has been deleted).

Let us see how the left shift operator can be used on the Fibonacci sequence.

We know that $\qquad\qquad\qquad F_{n+1} = L(F_n).$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

Also, $\qquad\qquad\qquad\qquad F_{n+2} = L(F_{n+1}) = L(L(F_n)) = L^2(F_n).$ $\qquad\qquad\qquad\qquad$ (6)

Now, $F_{n+2} = F_{n+1} + F_n$ yields $F_{n+2} - F_{n+1} - F_n = 0$. Rewriting this in terms of L and using (6) we get

$$L^2(F_n) - L(F_n) - F_n = 0.$$

'Factoring out' F_n (treating it as though it is a number) we get $(L^2 - L - 1)F_n = 0$. $\qquad\qquad$ (7)

This is an interesting property of the left shift operator L which allows us to treat it like a variable and to be able to manipulate it algebraically. In fact we can solve the factored version of the above equation separately.

We already know that the roots of the algebraic equation

$$x^2 - x - 1 = 0 \text{ are } \frac{1 \pm \sqrt{5}}{2}. \text{ Let } \theta = \frac{1+\sqrt{5}}{2} \text{ and let } \varphi = \frac{1-\sqrt{5}}{2}.$$

Hence to solve $(L^2 - L - 1)F_n = 0$ we solve its factored form $((L - \theta)(L - \varphi))F_n = 0$.

In other words, we need to solve $(L - \theta)F_n = 0$ and $(L - \varphi)F_n = 0$ separately and combine the results linearly.

Now $(L - \theta)F_n = 0$ implies $L(F_n) = \theta F_n = F_{n+1}$.

Further, $F_{n+1} = \theta F_n$ implies that $F_n = A\theta^n$ for some constant A. This is an example of a geometric sequence.

Similarly, solving $(L - \varphi)F_n = 0$ we get $F_n = B\varphi^n$ for some constant B.

Finally we obtain the general solution of $(L^2 - L - 1)F_n = 0$ as

$$F_n = A\theta^n + B\varphi^n \qquad\qquad\qquad\qquad\qquad\qquad\qquad (8)$$

This relation will help us find the formula for the nth Fibonacci number.

Substituting $n = 0$ in (8) and recalling that $F_0 = 0$, we get $A + B = 0$ or $B = -A$.

Substituting $n = 1$ in (8) and recalling that $F_1 = 1$, we get

$$1 = A\theta^1 + B\varphi^1 = A(\theta - \varphi) = A\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \sqrt{5}\,A.$$

This implies that $A = 1/\sqrt{5}$ and $B = -A = -1/\sqrt{5}$.

Plugging these values of A and B in (8), we have the much desired formula

$$Fn = \frac{1}{\sqrt{5}}(\theta^n - \phi^n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Simplifying we get, $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$

Let us verify if this works by substituting $n = 2$.

$$F_2 = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{1 + 2\sqrt{5} + 5 - 1 + 2\sqrt{5} - 5}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1,$$

which is true!

Conclusion

Fibonacci numbers appear in nature. They appear in the arrangement of leaves in certain plants, on the patterns of the florets of a flower, the bracts of a pinecone, and on the scales of a pineapple. There are numerous examples of Fibonacci numbers and the golden ratio in nature which students can explore in the form of a project. MS Excel can prove handy in exploring the patterns in the sequence, and this activity can be done with students at the middle school level. However, finding the general term of the Fibonacci sequence can be easily illustrated to secondary school students.

References

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