problem corner

Adventures in Problem Solving

Squares between Squares

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I. A 'Trivial' Observation

It is often the case — more often than one may expect— that in the successful solution of a problem, the seed of the solution is a very simple observation; so simple that we call it 'trivial', and we doubt its potential to play a significant role in the solution of any problem. We showcase such an instance here, starting with the following:

Observation 1.
Between two consecutive integers there does not lie another integer.

Stated this way the observation looks too trivial to be of use to anyone. But even simple observations have consequences! And these may well be of a less trivial nature than the original one. They in turn may give rise to other consequences, and so on, and these may progressively get less trivial. What we get in the end result can be significant and far from trivial!

For example, here is a consequence of the Observation 1, which we get by squaring the quantities:

Observation 2.
Let n be an integer. Between the perfect squares n^2 and $(n+1)^2$ there does not lie another perfect square.

From this we deduce the following, which looks more impressive than Observation 1:

Observation 3.
Let n and X be integers such that $n^2 \le X \le n^2+2n+1$. Suppose that X is a perfect square. Then either $X=n^2$ or $X=(n+1)^2$.

II. Applications

We make use of observations 2 and 3 made above in solving two problems posed in the 'Adventures' column of the November 2013 issue of *AtRiA*.

Problem 5, 'Adventures', November 2013
Find all integers n such that n^2+n+1 is a perfect square.

Solution Suppose that $n>0$. Then $n^2 < n^2+n+1 < n^2+2n+1$, so n^2+n+1 lies between the two consecutive squares n^2 and $(n+1)^2$ and thus cannot be a square. So there can be no solution with $n>0$.
Next suppose that $n<0$; then $n \le -1$, or $n+1 \le 0$.

Hence $n^2 + 2n + 1 < n^2 + n + 1 \le n^2$. So the only way for $n^2 + n + 1$ to be a square is to let $n + 1 = 0$, i.e., $n = -1$.

Since $n = 0$ also yields a square value, the possible n-values are $-1, 0$. Both these n-values yield $n^2 + n + 1 = 1^2$.

Problem 4, 'Adventures', November 2013 may be solved in the same way.

Problem 6, 'Adventures', November 2013

Find all integers n such that $n^4 + n^3 + n^2 + n + 1$ is a perfect square.

This is a substantially more complicated problem of the same genre, but it too can be solved the same way. The first challenge is to box the given quantity between two perfect squares. To do this we shall use the 'completing the square' reasoning yet again.

Recall that to add a term to $(x^2 + ax)$ so as to get a square, we *halve* the coefficient of x and use that to create the term: $x^2 + ax + (\frac{1}{2}a)^2 = \left(x + \frac{1}{2}a\right)^2$.

The same logic applied to $n^4 + n^3 + n^2 + n + 1$ suggests that the expression to be studied has to be $\left(n^2 + \frac{1}{2}n\right)^2$. For, when we expand this expression, we get both the n^4 and n^3 terms with the correct coefficients:

$$\left(n^2 + \frac{n}{2}\right)^2 = n^4 + n^3 + \frac{n^2}{4}.$$

Note that we have not got the right coefficient for n^2 (we have got $\frac{1}{4}$ instead of 1), and we haven't got the other two terms at all.

The fractions ($\frac{1}{2}$ and $\frac{1}{4}$) make things a bit awkward, so why don't we multiply everything by 4? That way, squares remain squares (for, if X is a square, then so is $4X$; this is crucial), and at the same time we get rid of the fractions. So we ask: *For which integers n is the quantity $A = 4\left(n^4 + n^3 + n^2 + n + 1\right)$ a square?*

So let us now box A between two consecutive squares. Since $2 \times \left(n^2 + \frac{1}{2}n\right) = 2n^2 + n$, the candidate squares are $B = \left(2n^2 + n\right)^2$ and $C = \left(2n^2 + n + 1\right)^2$. We now have:

$$\left(2n^2 + n\right)^2 = 4n^4 + 4n^3 + n^2,$$

$$\therefore\ A - B = 3n^2 + 4n + 4.$$

We shall show that this quantity is *always positive*. There are various ways of seeing this. The simplest is to compute the discriminant of the quadratic expression $3n^2 + 4n + 4$ using the well-known "$b^2 - 4ac$" formula. We get: $4^2 - 4 \times 3 \times 4 = 16 - 48$ which is negative. As the discriminant is negative, the expression $3n^2 + 4n + 4$ never changes sign. And since it is positive for $n = 0$, it is positive for all n.

(Another way to see that $A - B > 0$ is to write the expression for $A - B$ as:

$$\begin{aligned} A - B &= 3n^2 + 4n + 4 \\ &= n^2 + n^2 + \left(n^2 + 4n + 4\right) \\ &= n^2 + n^2 + (n+2)^2. \end{aligned}$$

Please complete this line of reasoning on your own.)

We see that $B < A$, strictly. Now let us look at $C - A$:

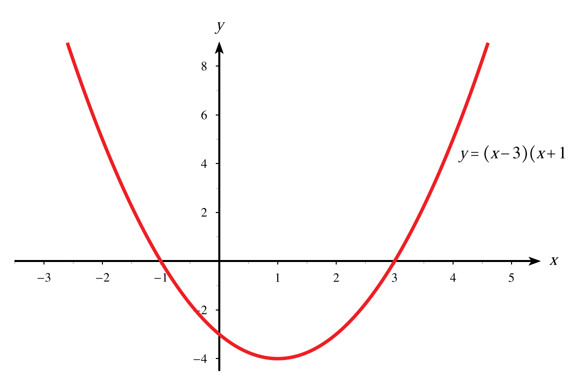
$$\left(2n^2 + n + 1\right)^2 = 4n^4 + 4n^3 + 5n^2 + 2n + 1,$$

$$\therefore\ C - A = n^2 - 2n - 3.$$

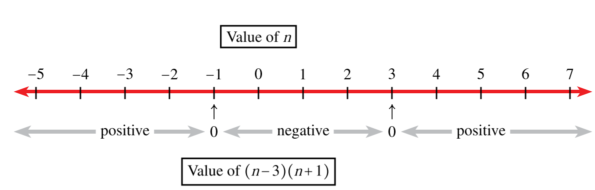
Conveniently for us, the form $n^2 - 2n - 3$ factorizes:

$n^2 - 2n - 3 = (n-3)(n+1), \quad \therefore C - A = (n-3)(n+1).$

The quadratic expression $(x-3)(x+1)$ gives rise to the following graph:



We see that the expression (n -3) (n +1) has the following sign profile:



Hence $C - A$ is positive if $n < -1$ or if $n > 3$.

This has important consequences:

Proposition 1. If either $n < -1$ or $n > 3$ then

$$(2n^2 + n)^2 < 4\left(n^4 + n^3 + n^2 + n + 1\right) < (2n^2 + n + 1)^2.$$

So if the integer n is less than -1 *or greater than* 3*, the quantity* $4\left(n^4 + n^3 + n^2 + n + 1\right)$ *lies strictly between two consecutive squares and is thus not a perfect square. Therefore the quantity* $n^4 + n^3 + n^2 + n + 1$ *is not a square if* $n < -1$ *or* $n > 3$.

Proposition 2. If $-1 \le n \le 3$ then

$$(2n^2 + n)^2 < 4\left(n^4 + n^3 + n^2 + n + 1\right) \le (2n^2 + n + 1)^2.$$

Equality holds in the inequality on the right side precisely when $n \in \{-1, 3\}$*. Therefore, the quantity* $4\left(n^4 + n^3 + n^2 + n + 1\right)$ *is a perfect square precisely when* $n \in \{-1, 3\}$.

We have fully solved the problem! The answer is: *The quantity* $n^4 + n^3 + n^2 + n + 1$ *is a perfect square precisely when* $n \in \{-1, 3\}$.
All this from the 'trivial' observation that between two consecutive integers there does not lie another integer! We have come a long way indeed. We invite you to apply these ideas to the following problem:

Find all integers n such that $n^4 + 2n^3 + 3n^2 + 4n + 5$ *is a perfect square.*

Remark 1. Here is a problem which closely resembles the one solved in the preceding pages: *Find all integers n such that* $n^3 + n^2 + n + 1$ *is a perfect square.*

But the resemblance is deceptive. Though the problem concerns a polynomial of lower-degree, it is significantly more difficult than the one we solved. Perhaps the main reason for this is that its degree is odd (3 rather than 4), so there is no easy way of boxing it between two squares. We shall not attempt to solve the problem here.

Remark 2. We mention here a second consequence of Observation 1 (but in another form); it is based on Problem 11121 from the well-known journal *American Mathematical Monthly*. For any positive integer n, we consider the set S_n of integers that lie strictly between n^2 and $(n+1)^2$. For example, $S_1 = \{2, 3\}$, $S_2 = \{5, 6, 7, 8\}$ and $S_3 = \{10, 11, 12, 13, 14, 15\}$. You can check that S_n has $2n$ elements. We now ask the following question:

Can we find two distinct numbers in S_n *whose product is a perfect square?*

The answer is clearly 'No' for $n = 1$ and $n = 2$ (please check). It turns out that the answer is 'No' for every n. The proof is elegant and instructive.

Proof We adopt the well-known approach of 'proof by contradiction'. Suppose that for some $n \ge 1$ there exist integers $a, b \in S_n$, $a < b$, such that ab is a square. Now any positive integer can be written as the product of a perfect square and a square-free integer (i.e., an integer not divisible by any perfect square larger than 1), simply by factoring out the largest possible perfect square from it. (For example, $20 = 2^2 \times 5$; $150 = 5^2 \times 6$.) Accordingly, we write $a = uv^2$ and $b = UV^2$ where u and U are square-free. Since ab is a perfect square, so is uU; but this implies that u and U are the same number! (Do you see why? It is because each prime factor in u and U occurs just once in each number.)

So we have $a = uv^2$ and $b = uV^2$ for some positive integers u, v, V with $v < V$. It must be that $u > 1$, else we get the chain $n^2 < v^2 < V^2 < (n+1)^2$, which is clearly absurd. Now from the relation $n^2 < uv^2 < uV^2 < (n+1)^2$ we get, by taking square roots:

$$\frac{n}{\sqrt{u}} < v < V < \frac{n+1}{\sqrt{u}}.$$

The difference between the numbers at the ends of this chain of inequalities is

$$\frac{n+1}{\sqrt{u}} - \frac{n}{\sqrt{u}} = \frac{1}{\sqrt{u}} < 1,$$

i.e., the difference is strictly less than 1. On the other hand, $V - v \ge 1$: the difference between the middle numbers is not less than 1. These relations contradict each other! Hence such a situation cannot happen. That is, integers a, b with the stated property do not exist.

III. Solutions to Problems 1–5 from the November 2013 Issue

1. (a) $3599 = 3600 - 1 = 60^2 - 1 = (60 - 1) \times (60 + 1) = 59 \times 61$

   (b) $8099 = 8100 - 1 = 90^2 - 1 = (90 - 1) \times (90 + 1) = 89 \times 91 = 89 \times 7 \times 13$

   (c) $4087 = 4096 - 9 = 64^2 - 3^2 = (64 - 3) \times (64 + 3) = 61 \times 67$

2. $x^4 + 4 = \left(x^4 + 4x^2 + 4\right) - 4x^2 = \left(x^2 + 2\right)^2 - (2x)^2 = \left(x^2 - 2x + 2\right) \cdot \left(x^2 + 2x + 2\right)$

3. *Find all integers n such that $n^2 + 10n + 20$ is a perfect square.*

   Let $n^2 + 10n + 20 = x^2$; then $n^2 + 10n + 25 = x^2 + 5$, so $(n + 5)^2 - x^2 = 5$, and:

   $$(n + 5 - x) \cdot (n + 5 + x) = 5.$$

   The factorizations of 5 are: 1×5, 5×1, $(-1) \times (-5)$, $(-5) \times (-1)$. Hence:

   $$(n + 5 - x, n + 5 + x) \in \{(1, 5),\ (5, 1),\ (-1, -5)\ (-5, -1)\}.$$

   It follows by addition that $2(n + 5) = 6$ or -6, so $2n = -4$ or -16, and therefore $n = -2$ or -8. Both these values of n correspond to the same value of x^2, namely: $x^2 = 4$.

4. *Find all integers n such that $n^2 + n$ is a perfect square.*

   If $n^2 + n$ is a square, so is $4(n^2 + n) = 4n^2 + 4n$. But $4n^2 + 4n + 1 = (2n + 1)^2$ is a square as well. So we have two squares differing by 1, which happens only with 0 and 1.

   (To see why the only squares differing by 1 are 0 and 1, let x, y be integers such that $x^2 - y^2 = 1$. Then $(x + y)(x - y) = 1$, hence $(x + y, x - y) = (1, 1)$ or $(-1, -1)$. The first possibility lead to $(x, y) = (1, 0)$, and the second one leads to $(x, y) = (-1, 0)$. Hence the two squares are $1^2 = 1$ and $0^2 = 0$.)

   Hence $n^2 + n = 0$, which yields $n = -1, 0$. So the answer is that the only square value taken by $n^2 + n$ is 0, which it takes when $n \in \{-1, 0\}$.

   Another approach. Note that $n^2 + n = n(n + 1)$. Since n and $n + 1$ are consecutive integers, they are coprime. Hence if their product is a square, both are squares or both are negatives of squares. Either way we get a pair of consecutive squares differing by 1. The only such squares are 0, 1. Hence it must be that one of $n, n + 1$ is 0. The two ways in which this can happen yield the solutions obtained above ($n = 0, -1$).

5. *Find all integers n such that $n^2 + n + 1$ is a perfect square.*

   This was solved above but we give another solution here.

   If $n^2 + n + 1 = x^2$, then $4n^2 + 4n + 4 = (2x)^2$, so $(2n + 1)^2 + 3 = (2x)^2$, which yields $(2x)^2 - (2n + 1)^2 = 3$. Hence $(2x - 2n - 1)(2x + 2n + 1) = 3$. Since $3 = 3 \times 1$, 1×3, $(-3) \times (-1)$, $(-1) \times (-3)$, it follows that $4n + 2 \in \{-2, 2\}$. This yields $n \in \{-1, 0\}$. So there are two integers n for which $n^2 + n + 1$ is a perfect square, -1 and 0. For both these n-values, $n^2 + n + 1$ takes the same value, which is $1 = 1^2$.

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