problem corner

Problems for the Middle School

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Problems for Solution

Problem III-1-M.1
Show that the following number is a perfect square for every positive integer n:

$$\underbrace{111111 \ldots 111111}_{2n \text{ digits}} - \underbrace{222 \ldots 222}_{n \text{ digits}}.$$

For example, $11 - 2 = 9$ and $1111 - 22 = 1089$ are perfect squares.

Problem III-1-M.2
On a digital clock, the display reads 6 : 38. What will the clock display twenty-eight digit changes later?

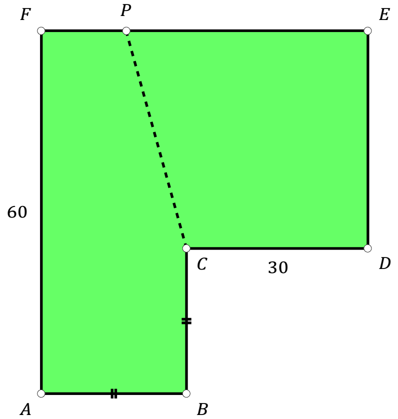
Problem III-1-M.3
The figure shows a hall $ABCDEF$ with right angles at its corners. Its area is 2520 sq units, and $AB = BC$, $CD = 30$ units, $AF = 60$ units. A point P is located on EF such that line CP divides the hall into two parts with equal area. Find the length EP.

Problem III-1-M.4
In a circle with radius 4 units, a rectangle and an equilateral triangle are inscribed. If their areas are equal, find the dimensions of the rectangle.

Problem III-1-M.5
Find the value of the following (no calculators!):

$$\left\lfloor \frac{2014^3}{2012 \times 2013} \right\rfloor - \left\lfloor \frac{2012^3}{2013 \times 2014} \right\rfloor.$$

Here the symbol $\lfloor\ \rfloor$ has the following meaning: if x is any real number, $\lfloor x \rfloor$ is the largest integer not greater than x. For example, $\lfloor 3.2 \rfloor = 3$, and $\lfloor -1.7 \rfloor = -2$. It is called the "greatest integer function".



Solutions of Problems in Issue - II - 3

Solution to problem II-3-M.1

Find the value of the following (no calculators!):

$$\frac{(2013^2 - 2019) \times (2013^2 + 4023) \times 2014}{2010 \times 2012 \times 2015 \times 2016}.$$

Let $a = 2013$; then $2019 = a + 6$, $4023 = 2a - 3$, etc., so the given expression equals:

$$\frac{\left(a^2 - a - 6\right)\left(a^2 + 2a - 3\right)(a+1)}{(a-3)(a-1)(a+2)(a+3)}$$

$$= \frac{(a-3)(a+2)(a+3)(a-1)(a+1)}{(a-3)(a-1)(a+2)(a+3)} = a + 1.$$

So the expression simplifies to 2014.

Solution to problem II-3-M.2

Can you find a pair of perfect squares that differ by 2014?

The answer is **No**. For suppose that $a^2 - b^2 = 2014$ where a, b are integers. Then $(a-b)(a+b) = 2014$. Since 2014 is even, at least one of the quantities $a - b$, $a + b$ is an even number. But $a - b$ and $a + b$ have the same parity (they are both odd or both even), so if one of them is even, then so is the other one. This means that the product $(a-b)(a+b)$ is a multiple of 4. However, 2014 is not a multiple of 4. Hence the given representation is not possible.

Solution to problem II-3-M.3

From a two-digit number n we subtract the number obtained by reversing its digits. The answer is a perfect cube. What could n be?

Let $n = 10a + b$ where a, b are digits. On subtracting its reversal, $10b + a$, we get the number $x = 9(a-b) = 3^2(a-b)$. For x to be a cube, $a - b$ would have to be 3 times a cube (this would lead to $x = 3^3 \times$ a cube). Since a and b are digits, the absolute value of $a - b$ cannot exceed 9. Hence if $a - b$ is 3 times a cube, it must be that $a - b = 3$ or -3. Therefore n is one of the following: 14, 41, 25, 52, 36, 63, 47, 74, 58, 85, 69, 96. For each of these, $x = \pm 27 = (\pm 3)^3$.

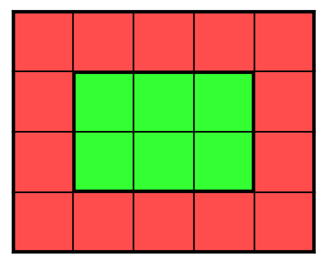
Solution to problem II-3-M.4

To a certain two-digit number m we add the number obtained by reversing its digits. The answer is a perfect square. What could m be?

Let $m = 10a + b$ where a, b are digits. On adding its reversal, $10b + a$, we get the number $y = 11(a+b)$. For y to be a square, $a + b$ would have to be 11 times a square (this would lead to $y = 11^2 \times$ a square). Since a and b are digits, the sum $a + b$ cannot exceed 18. Hence if $a + b$ is 11 times a square, it must be that $a + b = 11$. Therefore m is one of the following: 29, 92, 38, 83, 47, 74, 56, 65. For each of these, $y = 121 = 11^2$.

Solution to problem II-3-M.5

The rectangle shown has been divided into equal squares. The squares along the perimeter are shaded red; the rest of the squares are shaded green. Note that the number of red squares is greater than the number of green squares. What should be the dimensions of the rectangle if the number of red squares equals the number of green squares?



Let the dimensions of the rectangle be $a \times b$, with $a \geq b$. The inner rectangle (shaded green) has dimensions $(a-2) \times (b-2)$. For the areas of the red and green regions to be the same, the area of the green region must be half the area of the large rectangle, so we must have:

$$ab = 2(a-2)(b-2),$$

$$\therefore\ ab = 2(ab - 2a - 2b + 4),$$

$$\therefore\ ab - 4a - 4b + 8 = 0.$$

We must therefore find pairs (a, b) of positive integers that satisfy the equation $ab - 4a - 4b + 8 = 0$. The way we do this is based on factorization. (It is a fairly standard procedure.) It draws on the observation that $ab - 4a - 4b + 8$ is `almost' equal to the product $(a-4)(b-4)$, but not quite: we get 16 in place of 8. This prompts the following:

$$ab - 4a - 4b + 8 = 0,$$

$$\therefore\ ab - 4a - 4b + 16 = 8,$$

$$\therefore\ (a-4)(b-4) = 8.$$

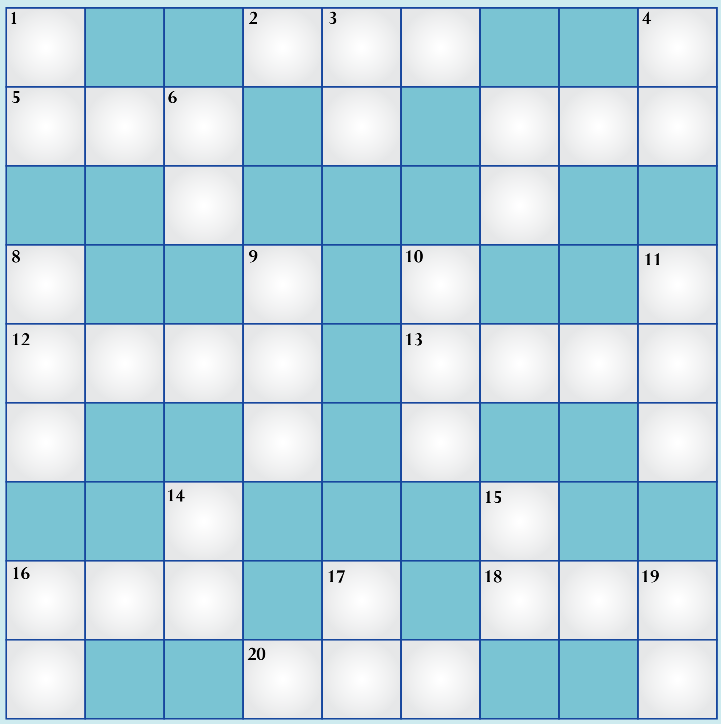
Hence $(a-4, b-4)$ are a pair of positive integers whose product is 8. In what ways can 8 be expressed as a product of two integers? The only ways are: 8×1 and 4×2. Hence $(a-4, b-4) = (8,1)$ or $(4,2)$, and therefore, $(a, b) = (12,5)$ or $(8,6)$. So the dimensions of the large rectangle are either 12×5 or 8×6. Observe that these satisfy the stated conditions:

$12 \times 5 = 60$, $\quad (12-2) \times (5-2) = 30$,

$30 =$ half of 60, $8 \times 6 = 48$, $(8-2) \times (6-2) = 24$,

$24 =$ half of 48.

number crossword-4



Clues Across :

1. Half of 16 A
3. The first digit is followed by its successor and then by its predecessor
5. The middle digit is the sum of the end digits
6. Area of a square of side 74
9. Digits in arithmetic progression
10. The square root of 417316
12. Two complete rotations and two degrees
13. Two centuries, two decades and two years

Clues Down :

1. 16A divided by 4D
3. One less than a positive multiple of 10
4. A dozen more than 19D
6. 14 D written in reverse
7. Twice the difference between 15D and 17D
8. Two and a half times 20A
9. A score of unlucky numbers
10. 3 D times the cube of 3
11. 9 times the second 3 digit prime.
14. A perfect square between 30 and 40
15. Square root of 12 A written in reverse
16. One day short of 10 weeks
17. 2 score and 2
19. One tenth of 9D