Problems for the Senior School

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problem corner

Problems for Solution

Problem III-1-S.1
Let $f(x) = ax^2 + bx + c$, where a, b, c are positive integers. Show that there exists an integer m such that $f(m)$ is a composite number.

Problem III-1-S.2
Show that the arithmetic progression $1, 5, 9, 13, 17, 21, 25, 29, \ldots$ contains infinitely many prime numbers.

Problem III-1-S.3
In $\triangle ABC$, the midpoint of AB is D, and E is the point of trisection of BC closer to C. Given that $\measuredangle ADC = \measuredangle BAE$, determine the magnitude of $\measuredangle BAC$.

Problem III-1-S.4
We know that a median of a triangle bisects it into two triangles of equal area. We also know that the medians of a triangle are concurrent. Given a $\triangle ABC$, does there necessarily exist a point D on side BC such that $\triangle ABD$ and $\triangle ACD$ have equal perimeter?

If such a point exists, then we can similarly obtain points E and F on AC and AB, respectively such that BE and CF bisect the perimeter of ABC. Are the lines AD, BE, CF concurrent?

Problem III-1-S.5
Let $A = 5^{2013}$ and $B = 4^{2013}$. Is $4^A + 5^B$ a prime number? Justify your answer.

Solutions of Problems in Issue-II-3

Solution to problem II-3-S.1
Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Find $P(3)$.

Putting $x = 1$ yields $P(0) + P(2) = 0$. Putting $x = -1$, we get:

$$P(-1) = P(0) - P(1) + P(2),$$

hence $P(1) = -1$. Putting $x = 2$, we get:

$$P(2) = P(0)-2-4P(0) = -2-3P(0) = -2+3P(2),$$

hence $P(2) = 1$ and $P(0) = -1$. Therefore $P(x) = -1 - x + x^2$ and $P(3) = 5$.

Solution to problem II-3-S.2
In $\triangle ABC$, the midpoint of BC is D; the foot of the perpendicular from A to BC is E; the foot of the perpendicular from D to AC is F; $BE = 5$, $EC = 9$; area of $\triangle ABC$ is 84. Find EF.

There are two cases: (i) E lies between B and C. (ii) E lies to the left of B on the line BC. Possibility (i) is depicted in Figure 1.

(i) We have $[ABC] = 84 = 1/2 \times 14 \times AE$, so $AE = 12$, $ED = EC - DC = 9 - 7 = 2$. Now assign coordinates: $E = (0,0)$, $A = (0,12)$, $B = (-5,0)$, $C = (9,0)$, $D = (2,0)$.

The slope of AC is $-4/3$, and the slope of DF is $3/4$. The equations of AC and DF are $x/9 + y/12 = 1$ and $y = (3/4)(x-2)$, respectively. Solving these two equations for x, y, we get $F = (162/25, 84/25)$. Hence $ED = \sqrt{(162^2 + 84^2)}/5 = 6\sqrt{37}/5$.

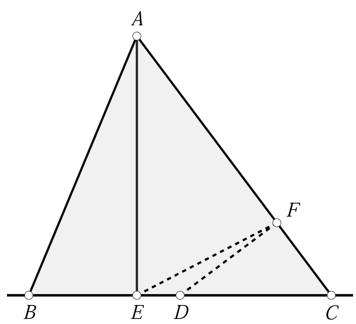

Figure 1.

(ii) Following the same steps we get $E = (0,0)$, $B = (5,0)$, $C = (9,0)$, $D = (7,0)$. The equations of AC and DF are now $x/9 + y/42 = 1$ and $y = (3/14)(x-7)$. Solving these we get $F = (1827/205, 84/205)$, so $EF = 21\sqrt{7585}/205$.

Solution to problem II-3-S.3

In how many ways can the numbers $-8, -7, -6, \ldots, 6, 7, 8$ *be arranged in a line so that the absolute values of the numbers do not decrease from left to right?*

If the stated property is to be satisfied, then for each $a \in \{1, 2, \ldots, 8\}$, the numbers $a, -a$ must occur together. Further, as the absolute values of the entries must not decrease, the numbers $1, 2, \ldots, 8$ must appear in this order, reading from left to right. If these two conditions are met, then the stated property will hold good. Having fixed such an arrangement, we observe that for each a, the number $-a$ must occur either immediately to the left or immediately to the right of a. Thus there are just two ways to insert $-a$ for any a. Hence the total number of legitimate arrangements is $2^8 = 256$.

Solution to problem II-3-S.4

Two ships sail with constant speed and direction. It is known that at 9:00 am the distance between them was 20 *miles; at 9:35 am,* 15 *miles; and at 9:55 am,* 13 *miles. What was the least distance between the ships, and at what time was it achieved? [IMO Short list, 1968]*

As the ships are sailing with constant speed and direction, the second ship is sailing at a constant speed and direction with reference to the first ship. Let A be the constant position of the first ship in this frame, and let ℓ be the path of the second ship in relation to the first one. Let points B_1, B_2, B_3, and B on ℓ be positions of the second ship with respect to the first ship at 9:00, 9:35, 9:55, and the moment the two ships are closest to each other. Then we have the following equations: $AB_1 = 20, AB_2 = 15, AB_3 = 13$; $B_1B_2 : B_2B_3 = 7 : 4$, $AB_i^2 = AB^2 + BB_i^2$.

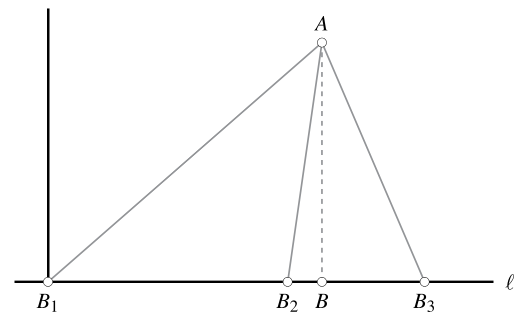

Figure 2.

We now adopt coordinates (see Figure 2). Let ℓ be taken to be the x-axis, with $B_1 = (0,0)$, $B_2 = (7c, 0)$, $B_3 = (11c, 0)$ where $c > 0$; it is assumed that the second ship is moving along the x-axis in the positive direction; let $A = (a, b)$. Then we have:

$$a^2 + b^2 = 20^2,$$

$$(a - 7c)^2 + b^2 = 15^2,$$

$$(a - 11c)^2 + b^2 = 13^2.$$

These yield, by subtraction:

$$14ac - 49c^2 = 20^2 - 15^2 = 175,$$

$$22ac - 121c^2 = 20^2 - 13^2 = 231.$$

Treating these as a pair of simultaneous equations in ac and c^2 we get $ac = 16$, $c^2 = 1$, hence $c = 1$, $a = 16$. This in turn yields $b = \pm 12$. Hence the closest distance of A from ℓ is 12 miles, and the time at which this takes place is 16×5 minutes after 9:00, i.e., at 10:20 am.