# Of Monsters and Moonshine

## A review of 'Symmetry' by Marcus Du Sautoy

review

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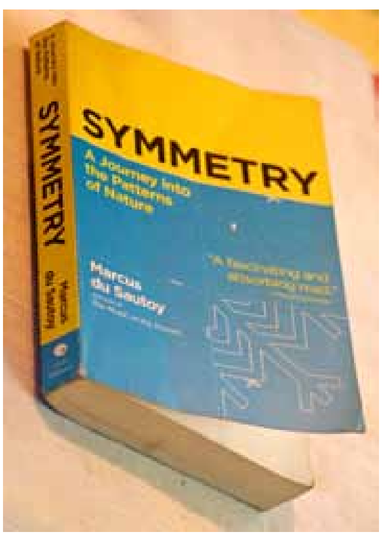



Symmetry is a topic that resonates with audiences of varied backgrounds and levels of mathematical knowledge. One can ask a layperson to explain what symmetry means to them and there would invariably be a fairly accurate response from an informal or non-mathematical point of view. However, symmetry has deep roots in mathematics and in some sense pervades most areas of study in mathematics. The mathematical study of symmetry though has its primary residence in an area of abstract algebra called 'group theory'. What is remarkable about Marcus Du Sautoy's book on symmetry is that the mathematics underlying the study of symmetry is explained without recourse to technical mathematical language. While the word 'group' makes its debut on page 9 of the book, it is only much later that its mathematical context is explained. By then the reader has had sufficient foundation laid to absorb the mathematical context.

***Keywords:*** *symmetry, patterns, Marcus du Sautoy, reflection, rotation, tiling, group, permutation*

As a mathematician there was a special joy in realising that it was possible to talk about areas of research in a language that would find consonance with the interested reader. The book begins with notions of symmetry that are commonplace or intuitive notions. Through the course of the book the reader is taken on a journey that explores the connections of symmetry with nature, evolution, psychology, music and even mathematics at the research level. There is also a conscious attempt to illustrate mathematical ideas with concrete examples from everyday life.

The book starts with the author on the shore of the Red Sea contemplating the fact that he has turned 40. The number 40 is important in a mathematician's life. The Nobel Prize equivalent in Mathematics is the Field's medal. In a way it is tougher to get a Field's medal than a Nobel Prize as at most four are awarded every four years and only to mathematicians who have done outstanding work and have not yet attained the age of 40.

The chapters in the book traverse the months of a calendar year beginning with the first chapter titled *August: Endings and Beginnings* and finishing with *July: Reflections.* Thus the book intertwines a year of Marcus Du Sautoy's life, his forays into searching for symmetrical objects that are part of his research, his encounters as a mathematician with 'symmetry seekers'; a term used for the mathematicians trying to classify and quantify 'indivisible collections of symmetry', with the story being told of the main protagonist, namely, symmetry.

Since a year of Marcus's life is interlinked with the story of symmetry we learn about how the author got interested in mathematics at the age of 12 because of a schoolteacher who encouraged him. This might indeed be the case for many a mathematician. A book the author was encouraged to read, as a schoolboy was *The Language of Mathematics* by Frank Land. If one thinks about it with some care, mathematics as a language is particularly efficient in expressing precisely and concisely the statements that one wishes to make. The problem though is that it is not always an easy language to master.

There are several intersecting strands that are covered in the book. One strand represents the usual story that one expects while learning about symmetry: reflections and rotations of regular geometric figures like the square, equilateral triangle to those of the five platonic solids, symmetries of infinite figures like wall-paper patterns and tilings. The chapter *October: The Palace of Symmetry,* discusses the search by the author and his son Tomer for the 17 wallpaper patterns or tessellations that exist in the Alhambra Palace in Granada, Spain, built around 1300 by Spain's Muslim rulers. Another strand brings to fore the life histories and works of the mathematicians of the Renaissance period leading to those from the early 19th century who were responsible for creating the mathematical language of group theory to analyse symmetries. These stories are the fodder for Chapters 5-8 from *December: connections* to *March: indivisible shapes.*

*April: Sounding Symmetry,* as the title hints at, discusses the links between western classical music and symmetry. It also points out the opposing philosophy, between when musicians use symmetrical object as a basis for creating their music but keep them secret from the audience, and the task undertaken by a mathematician of laying bare all the facts logically about the objects of study.

The strand where the book goes beyond the expected is when it moves into the difficult territory of describing one of the mammoth tasks that occupied the symmetry seekers for a large part of the previous century, namely, 'the classification of simple finite groups'. This history is explained entirely in terms of 'indivisible symmetry groups' with many anecdotes and mathematical experiences thrown in. While this thread is woven into several chapters, the last three are primarily devoted to this twentieth century tale.

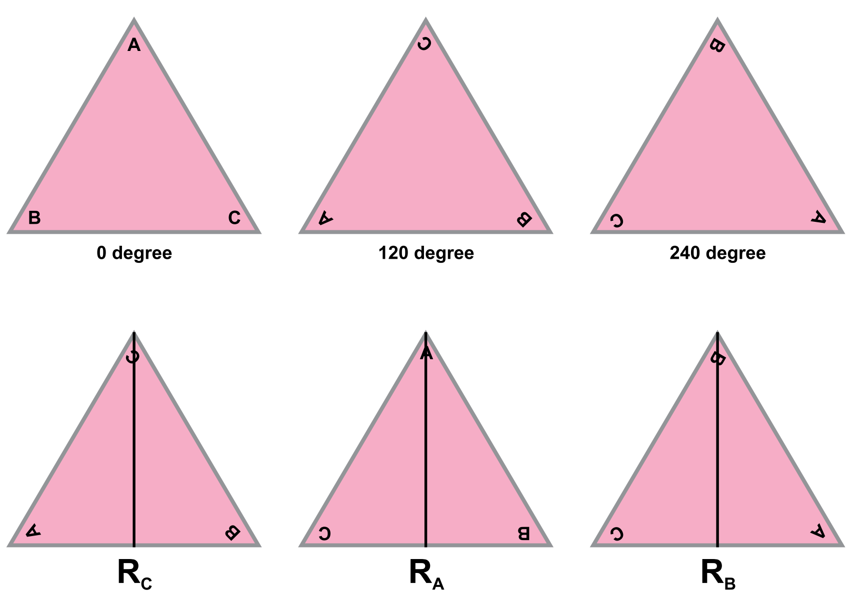
There are strands that are entirely missing, though. The book is deficient in telling the story of symmetry of non-western cultures. The Asian experience with symmetry finds no place. Indeed there is hardly any mention of the role played by symmetry in ancient or even medieval India. It is a lacuna that is not even acknowledged in passing by the author. Euro-centrism is the lens used.

Let us think of symmetry as 'a magic trick move' that keeps an object looking exactly like it did and in the exact same position as it occupied to start with. For example, as explained in the book, if we take a 50 pence coin (which is shaped like a regular heptagon or seven-sided figure) and draw its outline on a piece of paper, then a symmetry of the coin is any move or action that can be performed on the coin which brings the coin back into the outline drawn. In other words if someone had closed their eyes while the symmetry was being performed on the coin then they would assume that nothing had taken place. If we forget the markings on the coin and use this definition of symmetry then it is not too difficult to see that a regular heptagon has 14 symmetries in all.

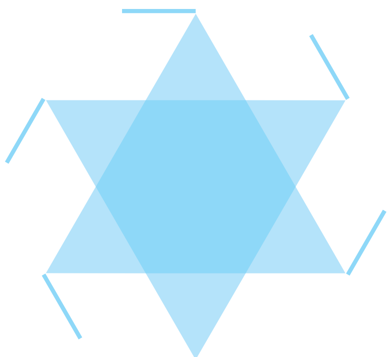
An easier example to work with is the equilateral triangle. It has six symmetries. Three are mirror symmetries or reflections, each about a line joining a vertex to the mid-point of the opposite side. These three lines of reflection meet at a point that is like the 'centre' of the equilateral triangle. The other three symmetries of the equilateral triangle are rotations through $0^\circ$, $120^\circ$ and $240^\circ$ respectively, say in the counter-clockwise direction, about an axis passing through the centre and perpendicular to the plane of the triangle. For example, if the three vertices of an equilateral triangle are marked A, B, C in the counter-clockwise direction then the $120^\circ$ counter-clockwise rotation will take A to B, B to C and C back to A. The figures show the effect on the vertices after a reflection or rotation symmetry has been performed.

A six-pointed star with no reflection symmetries (see figure) also has exactly six symmetries; these are rotations through $0^\circ$, $60^\circ$, $120^\circ$, $180^\circ$, $240^\circ$ and $300^\circ$. Here too, we can keep track of the symmetries by assuming a marking of the vertices and noting the effect of the respective symmetries on the vertices.

But is the collection of six symmetries of an equilateral triangle the same as the collection of six symmetries of a six-pointed star with no mirror symmetries? The answer is No: the two collections of symmetries are not the same when we consider them as 'groups'. Here is one way to see this: for the six-pointed star, if we take any two symmetries, it does not matter in which order we apply them; the final result is the same. But in the case of the equilateral triangle, if we apply a reflection followed by (say) a $120^\circ$ rotation, we get a different symmetry than when we first apply a $120^\circ$ rotation followed by the reflection. If we let the symbol $M$ denote the reflection and $R$ the rotation through $120^\circ$, then in the language of mathematics we write: $M * R \neq R * M$. (The symbol $M * R$ denotes that $R$ is applied first and then $M$.)



The point to note is that for any given object, one symmetry followed by another one leads to yet another symmetry of the figure. For example, the six symmetries of the equilateral triangle form such a closed system. In other words, if $R$ and $S$ are symmetries of the object, then so is $R * S$. It can be checked that if $R$, $S$ and $T$ are symmetries of the same object, then $R * (S * T) = (R * S) * T$.



This is called the 'associative property of $*$'. Every figure has the 'do-nothing' or $0^\circ$ rotation. If we denote it by $E$, then we have $R * E = R = E * R$ for all symmetries $R$ of the given object. The symmetry $E$ is called the 'identity'. Also, for every symmetry $R$ of the figure there is a reverse move which we denote by $R'$; this has the property that $R * R' = E = R' * R$. The symmetry $R'$ is called the 'inverse' of $R$. For example, the inverse of the $120^\circ$ rotation of the equilateral triangle is the $240^\circ$ rotation.

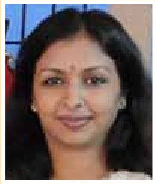
Such a closed system as described above is a *group*. The collection of all symmetries of any object is always a group, usually referred to as its *group of symmetries*. The young French mathematician Évariste Galois discovered this in 1829 at the age of 17. He came upon the idea while investigating something that, on the surface, seemed far removed from symmetry of objects; namely, his investigations into solutions of polynomial equations of the fifth degree or higher. While Galois did not quite see the shapes hiding in the solutions of these equations, he was able to associate with each equation a 'group of permutations' of its solutions. He showed by analysing these groups that a certain property of the group would decide whether one could express the solutions of the equation in terms of the constants occurring in the equation, rather like we can for a quadratic equation (and indeed for a cubic and a quartic).

The discerning reader may recognise that integers under addition also form a group. The prime numbers play a very important role amongst natural numbers. We learn at school that any counting number greater than one can be written as a product of primes. The primes are the counting numbers with exactly two factors, 1 and the number itself. Thus, primes are like the building blocks for the natural numbers. In much the same way, with the development of group theory in the 19th century, mathematicians realised that every symmetrical object could be built from smaller objects whose collection of symmetries were indivisible. For example, the 15 rotations of a regular 15-sided figure can be built from the 5 rotations of a pentagon and 3 rotations of an equilateral triangle. The two groups of 5 rotations of a pentagon and 3 rotations of an equilateral triangle each constitute an indivisible collection (group) of symmetries, in the sense that they cannot be built up from objects with a strictly smaller collection of symmetries. Such groups are also called 'simple groups'.

In the second-half of the twentieth century a prodigious task was undertaken to create a 'periodic table' of simple groups. There were a large number of group theorists working in all corners of the globe in a concentrated effort to find and classify all finite simple groups. The commander in chief of this project was the American mathematician, Daniel Gorenstein. The thrill of the chase, the discoveries of very large finite simple groups, christened 'baby monster' and 'monster', the story of how mathematicians involved in this esoteric quest managed to more-or-less complete the Herculean task is a story of moonshine and monsters and is narrated very well, with personal anecdotes involving the author and colourful descriptions of the mathematicians involved. As an example the following is the author's description of the Cambridge mathematician Simon Norton who was a contributor to the quest: "I could see what looked like a tramp, with wild black hair sprouting out all over his head, trousers frayed at the turn-ups, wearing a shirt full of holes. He was surrounded by plastic bags which seemed to contain his worldly possessions. He looked like a scarecrow."

The research questions, which the author is engaged in answering, are also discussed. Some of these are: How many objects are there with a prime power number of symmetries? How does one build symmetrical objects living in 'higher' non-visible dimensions from say the group of rotational symmetries of an equilateral triangle?

Marcus du Sautoy, as the blurb at the back of the book tells us, is the 'Charles Simoyni Professor for the Public Understanding of Science' at the University of Oxford and with this book he more than satisfies the job description. At the heart of his writing is the genuine wish to communicate the joy of doing and thinking mathematics.
The style and substance of his writing conveys this. The book is laid out with a structure that interweaves a year in the life of the author with his quest to demystify symmetry and at the same time make the case for symmetry as an all pervasive ideal that the human brain seems to be attracted to and finds hard to disengage from. The book comes highly recommended by this reviewer.



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