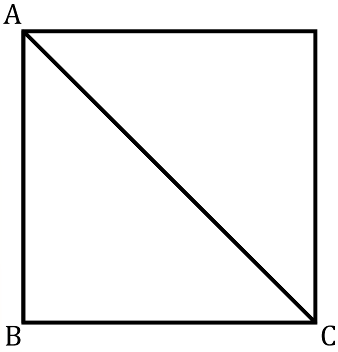
Letter to the Editor

In "An 'Origamics' Activity: X-lines" (AtRiA, November 2013) , the following result had been stated: A point is taken on the top edge of a square sheet of paper, and the bottom corners are folded to the point, generating two straight lines. The points of intersection of the X-creases fall on the vertical midline. The reader was asked to prove this observation. Here is a proof.

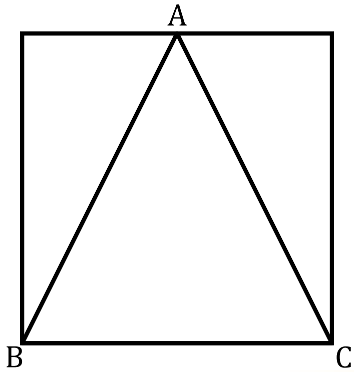
Let A be the point on the top edge, and let B, C be the two bottom corners. The fold produced by bringing B to A is the perpendicular bisector of segment AB, and the fold produced by bringing C to A is the perpendicular bisector of AC. So AB and AC are the perpendicular bisectors of sides AB and AC of triangle ΔABC, while the perpendicular bisector of side BC is the vertical midline! These three lines concur, so the point of intersection of the X-lines will lie on the vertical midline. So the point of intersection is nothing but O, the circumcentre of ΔABC.

This proves the third observation: "The distances from the point of intersection to the starting point and to each of the lower vertices are equal" (each distance is the circumradius of ΔABC).

Now take the second observation. The points of intersection of the X-creases lie along the midline and **lie below the centre of the square within a certain range** (emphasis added). Let us look at O as the point of intersection of the perpendicular bisectors of AB and BC (the vertical midline).The perpendicular bisector of AB intersects AB at its midpoint, which lies on the horizontal midline, i.e., the same level as the centre of the square. The slope of AB is positive, implying that the slope of its perpendicular bisector is negative. O lies on this line and to the right of the midpoint of AB. So O lies below the midpoint of AB, i.e., below the horizontal midline, hence below the centre of the square.



O does touch the centre of the square when A is at either of the top corners. When A is at the top left corner, we get a right triangle ABC with $\angle B = 90°$ and O is the midpoint of the hypotenuse AC. AC is a diagonal of the square, so its midpoint O is the centre of the square.



O reaches the other extreme, i.e., the lowest point, when A is at the middle of the top edge, i.e., the midpoint. Then AB has the lowest possible slope, hence its perpendicular bisector dips most steeply and cuts the vertical midline at the lowest possible point. Then O is at a distance 3/8 BC from BC. [This can be calculated considering $\angle BOC = 2\angle BAC$ and $r + r\cos\angle BAC$ = height of ΔABC = BC (since the paper is square), where r is the circum radius of ΔABC.]

As A moves from the top left corner to the midpoint of the top edge, O moves from the centre of the square to the above height. So far AB < AC, and we end with AB = AC. Then O reverses its trip back to the centre of the square as A continues from the midpoint of the top edge to the top right corner. Here AB > AC; they have switched roles, and everything now repeats in the reverse order.

Note: We don't need to restrict ourselves to a square sheet for this activity. It works equally well for a rectangular sheet. The lower extreme of the range would need to be recalculated given the ratio of height of the paper (or ΔABC) to BC, and in case BC becomes longer than twice the height of ΔABC, something interesting but inconvenient happens. It is left to the curious reader to explore this!

Contributed by

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