A thing of beauty …

Covering the Plane with Repeated Patterns - Part II

… is a pattern forever

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In Part I of this article (reference 1) we had noted how some regular polygons fit with each other to cover the plane without either gaps or overlaps, in arrangements called tilings. During our bus tour around the historic monuments of Delhi (described in the same article), we had seen many patterns based on simple rules, resulting in intricate tilings with great aesthetic appeal. Such patterns have been of interest to humans from ancient times, perhaps dating to the time when we started making shelters and used the logic of fitting rocks and weaving leaves to cover space while minimizing gaps. Over time, such endeavours took on artistic forms. Societies made use of tiles and patterns to emphasize different aspects of their culture. For example, Romans and some Mediterranean people portrayed human figures and natural scenes in their mosaic; the artistic impulse of the Arab artisans showed in their use of shape and colour to build complex geometric designs (as seen in the tiling patterns at Red Fort, Jama Masjid, Qutab Minar and Chandni Chowk; the Alhambra Palace in Granada, Spain, is another rich site of such patterns). Now, in Part II, we look at simple ways by which regular tessellations can be modified to make appealing patterns. We use simple techniques of colouring, shading or modifying a polygon to make interesting designs. The examples taken in this article are basic but lead to many possibilities that the readers can explore.

Keywords: *Pattern, tessellation, tiling, symmetry, art, architecture*

Art work using modified tessellations

We now revisit the regular tessellations made of congruent copies of one kind of regular polygon: equilateral triangles, squares and regular hexagons. An examination of their properties and their symmetries will help us deconstruct and recast the polygons to produce irregular motifs that will tile. We shall see that the art and architecture of tiling needs technical skills and imaginative ideas.

Modifying the regular quadrilateral

We start with the square. This shape has many symmetries that help us make irregular tessellating units. Some basic properties of this shape are: it has two pairs of equal, opposite and parallel sides; its adjacent sides are equal; it has rotational symmetry of order 4. We will now use combinations of these properties to modify our fundamental square tile.

To create the modified tile, carve out a piece along the length and translate it to the opposite parallel side. To make the design more interesting, chop off another piece along the width and translate it to the side opposite it. The pattern in Figure 1(a)-1(b) is based on translational symmetry of a square.

Figure 1(a)
The modified shape tessellates with translation symmetry. When embellished with some imaginative designs,the effect is enhanced.

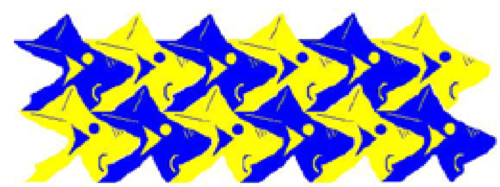

Figure 1(b)
Likewise, motifs can built on pairs of adjacent sides of a square to create tiles that show four-fold rotation symmetry, akin to that of the parent tile. Figures 1(c)-1(e)are some such patterns.

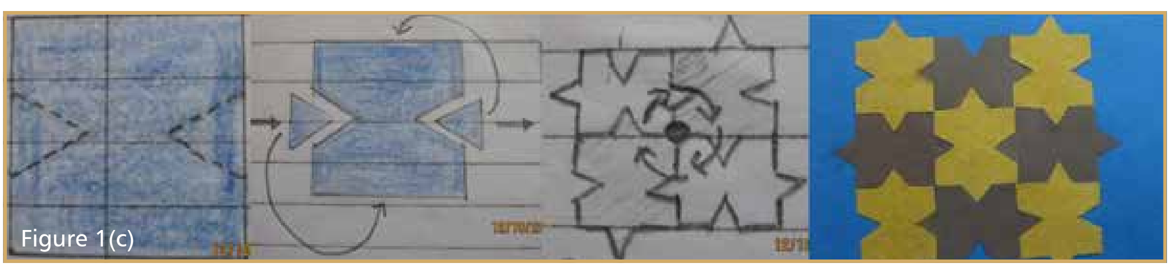

Figure 1(c)

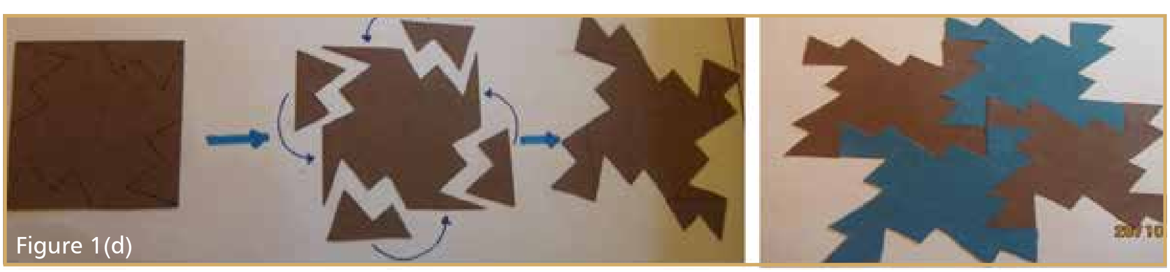

Figure 1(d)

Figure 1(e)

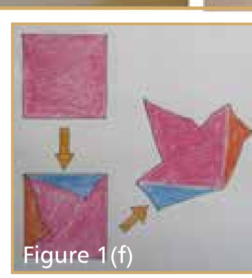

Figure 1(f)

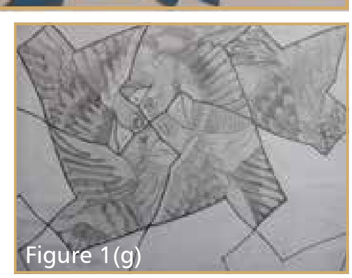

Figure 1(g)

By ornamenting the modified tile produced from a square (Figure 1(f)) with some enterprising art work, we get an interesting bird pattern (Figure 1(g)).

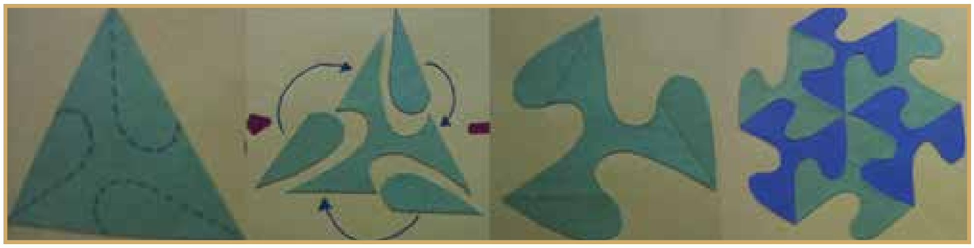

Figure 2(a)

Figure 2(b)

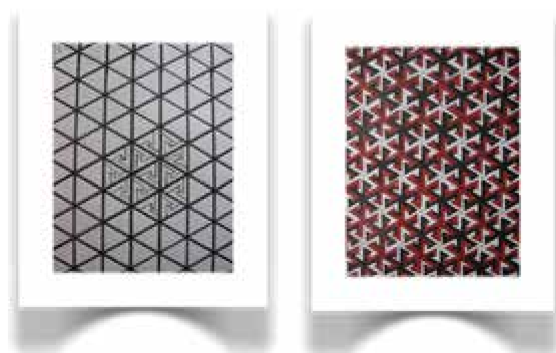

Figure 2(c)

Figure 3(a)

Modifying the regular triangle

The second set of Euclidean tessellations can be created from equilateral triangles. In this section we will explore one of the ways that was used to modify an equilateral triangle (Figure 2(a)-2(b)).

Draw curved lines inside the triangle, from a vertex to the midpoint of a side. Carve out a section with this as a boundary, give it a half-turn about the midpoint, and paste the piece on the outside of the other half length. Repeat this on the other two sides. Use the resulting tile as a stencil and create more tiles. We can now fit these tiles with one other, giving us a tessellation with rotational symmetry like that of the parent tile (the tessellation extends to the entire plane).

Modifying the regular hexagon

Regular hexagons can also be modified using their pairs of equal sides. The modification shown in Figure 3(a) is self-explanatory.

Decoding the tile

An interesting exercise with tessellations is to deconstruct the modified tile and to work out what must have been the parent tile. To do this, you will have to do the reverse of what was described above. Let's begin with a commonly used tile, often seen on the pedestrian paths of Delhi Metro stations (Figure 4(a)). Note that at any vertex three tiles interlock with each other, presenting the same rotational symmetry as that of a regular hexagon. By doing some basic modifications on the pairs of opposite and parallel sides, we can reshape it to get the desired motif.

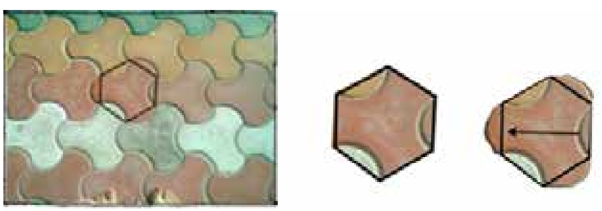

Figure 4(a)

Figure 4(b) is trickier. We see that at each vertex three of the lizards meet, exhibiting three-fold rotational symmetry. Now we must untangle the shape. To do this, trace out the irregular shape (the lizard) and circumscribe it with the associated regular polygon (here it is a hexagon). Now you must identify the portions sliced off and shift them to get the desired shape. There is no loss of any portion, so the area of the modified tile is the same as that of the parent tile.

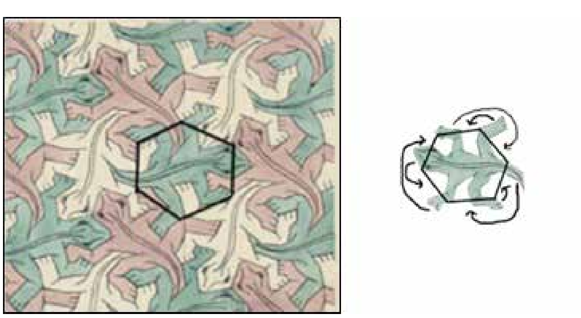

Figure 4(b)

In Figure 4(c) we see four fish interlocking at each vertex, showing rotational symmetry of order 4. It is thus easy to decode that the parent shape must have been a square.

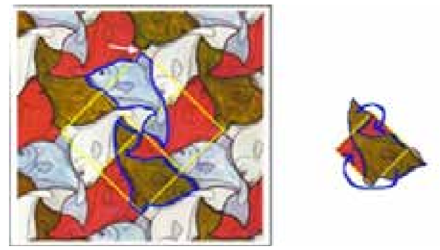

Figure 4(c)

The tessellation in Figure 4(d) was produced using equilateral triangles. Note the art work!

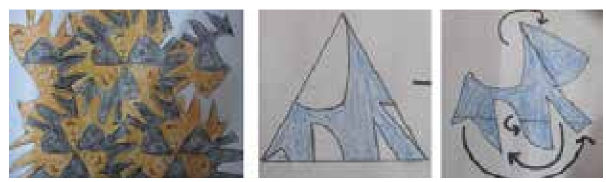

Figure 4(d)

Time for the thinking cap: some exercises

Can you guess the parent tile that has been modified to make the given tessellations (Figure 5(a)-5(e))? What properties of the parent tile helped in the modifications?

Figure 5(a)

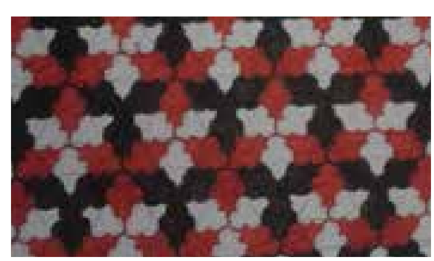

Figure 5(b)

The best way to deconstruct these tessellations is to use tracing paper to draw the outline of the repeating unit. By placing the traced figure on its look-alike, we deduce the underlying symmetry. The symmetries then help reveal the parent shape and the modifications done to it.


Figure 5(c)


Figure 5(d)


Figure 5(e)

Exploring further

Tessellations offer a huge range of mathematical explorations. One can only scratch the surface in a short article like this one. Some topics we have not explored relate to tessellations in which the units do not maintain the kind of repetition that equilateral triangles, squares and regular hexagons show. For example, there are tilings based on non-regular pentagons. Modifications in these polygons also make for intricate and artistic designs (Islamic star and the Penrose patterns) and can be explored further.

Middle grade teachers can use some of these examples for introductory work in tessellations. Concepts such as angle, area, perimeter, symmetry, closest packing, inscribed and circumscribed circles and more can be integrated and consolidated through such project work.

Further Reading

i. Grünbaum, B. & Shephard, G. C (1986), *Tilings and Patterns.* This is a comprehensive text on tessellations. It is a rich source of ideas that can be integrated in school geometry.

ii. Steinhaus, H. (1999), *Mathematical Snapshots* (Dover). Elementary yet engrossing.

iii. Critchlow, K (1970), *Order in space: A Design Source Book,* New York: Viking Press

Acknowledgements

The photographs have been taken by me and my B.Ed. students (batches of 2012-13 and 2013-14). I acknowledge my students' efforts in making intricate tessellations and extend my thanks to them for allowing me to use their designs for this article. Some pictures were taken from open sources such as Wikipedia.

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