The art …

A Proof Without Words for Viviani's Theorem

…of mathematics

feature

Readers of this magazine may recall that in the December 2012 issue we dwelt on Viviani's theorem and considered different ways of proving it. In the process we studied a few extensions and generalizations of the theorem. In this article we return to this theorem and discuss a proof-without-words (PWW) for the result. See [3] for the original version of the proof.

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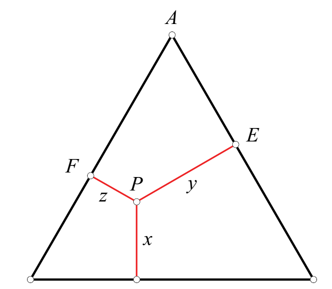


Figure 1. Viviani's theorem: $x + y + z =$ constant

Here is the statement of the theorem. Let $\triangle ABC$ be equilateral, and let P be any point in its interior (Figure 1). Then: *The sum of the distances from P to the sides of the triangle is a constant*. Thus, if perpendiculars PD, PE, PF are drawn from P to the sides BC, CA, AB, and their lengths are x, y, z, respectively, the sum $PD + PE + PF = x + y + z$ is the same for all positions of P.

Keywords: *Viviani, proof without words, equilateral triangle, translation, congruence, altitude*

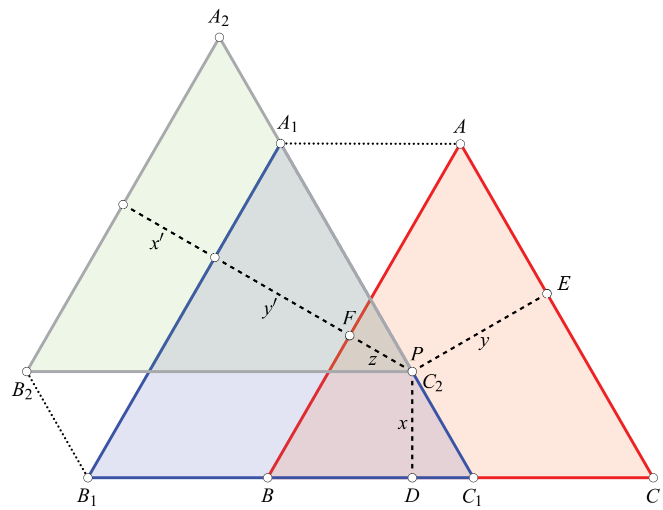


Figure 2. PWW for Viviani´s theorem. Note that $P = C_2$

In Figure 2 we see the proposed PWW. We start by translating $\triangle ABC$ along line BC to position $A_1B_1C_1$ so that point P lies on side A_1C_1. Then we translate $\triangle A_1B_1C_1$ along line A_1C_1 to position $A_2B_2C_2$ so that P coincides with vertex C_2. By construction, therefore, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are congruent copies of $\triangle ABC$, and $A_2A_1C_1$ is parallel to AC, and B_2C_2 is parallel to B_1BC_1C. From P we drop a perpendicular to A_1B_1 and A_2B_2. Let the distance between the parallel lines AB and A_1B_1 be y', and let the distance between the parallel lines A_1B_1 and A_2B_2 be x'. (The symbols y' and x' are marked in the figure.)

Since parallelograms ABB_1A_1 and ACC_1A_1 are congruent to each other, the distances between corresponding pairs of opposite edges are equal; hence $y' = y$.

Similarly, since parallelograms $B_1C_1C_2B_2$ and $B_1A_1A_2B_2$ are congruent to each other, $x' = x$. Therefore $x + y + z = z + y' + x'$.

But $z + y' + x'$ equals the height of $\triangle A_2B_2C_2$, which is the same as the height of $\triangle ABC$. It follows that $x + y + z$ is equal to the height of $\triangle ABC$.

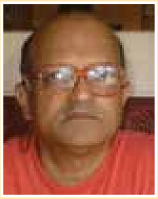
For another PWW of Viviani's theorem, see [1]. [Note from the editor: This issue of *At Right Angles* has a separate article on the subject of PWWs.]

References

[1] Kawasaki, K. *Proof Without Words: Viviani's Theorem*. https://www.maa.org/sites/default/files/3004415840629.pdf.bannered.pdf

[2] Bogomolny, A. *Viviani's Theorem: What is it?*. From "Interactive Mathematics Miscellany and Puzzles". http://www.cut-the-knot.org/Curriculum/Geometry/Viviani.shtml, Accessed 18 November 2014

[3] Nelsen, Roger B. *Proofs Without Words: Exercises in Visual Thinking*, page 15. MAA



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