Simple formulas …

Approximating Square Roots and Cube Roots

Do they work?

The idea of equivalent geometric forms is used in this study to devise simple formulas to estimate the square root and the cube root of an arbitrary positive number. The resulting formulas are easy to use and they don't take much time to calculate. They give excellent estimates which compare very favourably with those given by a pocket calculator.

ALI IBRAHIM HUSSEN

Introductory Remarks

Historically the Babylonians were the first to devise an iterative method for computing square roots of numbers using rational operations that are easy to carry out ([1], [2]). The ancient Chinese method (200 BCE), commented on by Liu Hui in the third century CE, is similar in procedure to the long division method used in schools even today ([2]). The Greek mathematician Heron of Alexandria, who gave the first explicit description of the Babylonians iterative method, also devised a method for cube root calculation in the first century CE ([3]). In 499 CE, Aryabhata (Indian mathematician and astronomer) gave a method for computing cube roots of numbers of arbitrary size ([5]). In the sixteenth century, Isaac Newton devised an iterative method used to calculate square and cube roots of arbitrary numbers.

The method used in this article is different from the methods mentioned above. It is based on the idea of changing a geometric

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figure or form into an equivalent 'regular' one with equal area or volume, e.g. changing a rectangle into a square, or a rectangular cuboid into a cube.

The idea was used by ancient Indian engineers who worked with designing and constructing temples (600 BCE). They used a practical method for changing a rectangle into a square. The method appears in the *Sulba-sutras* by Baudhayana (Sulba-sutras means the 'Rule of the Chord') ([6]). The idea is used also by David W. Henderson in his attempt to solve quadratic and cubic equations geometrically ([7]). In our case an algebraic analysis is performed instead of a geometric one.

1. Estimation of Square Roots

Let n be the given number whose square root is required. (Here, of course, n is positive.) We start by expressing n as a product ab, with a as close to b as possible (the closer the better). The task of computing $\sqrt{n} = \sqrt{ab}$ may be expressed geometrically as: Construct a square whose area is equal to that of a rectangle with dimensions $a \times b$ (Figure 1).

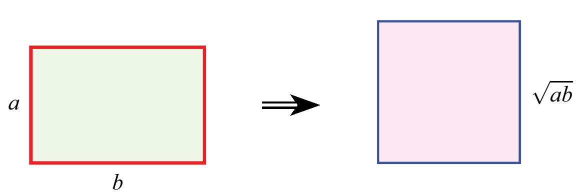

Figure 1.

Now note that $\sqrt{ab}$ is the *geometric mean* (GM) of the quantities a and b. We may try to approximate the GM by the *arithmetic mean* (AM) of a and b. It is well known that the AM exceeds the GM (strictly) if the numbers are unequal. But if the numbers are close to one another, then the AM is quite close to the GM.

Geometrically this may be viewed as follows. Assume that $a < b$. Within the $a \times b$ rectangle we draw a square on the shorter side a, then slice the leftover portion into two equal halves and stack them (one each) on to two adjacent sides of the square (Figure 2). The resulting shape is a square of side $\frac{1}{2}(a + b)$ minus a small piece in the corner, which is a square of side $\frac{1}{2}|a - b|$. If a is close to b, then this portion will be very small. So if we ignore the little square, the larger square may be taken to be the one we seek.

First approach to approximating square roots. The above reasoning gives us our *first algorithm* for approximating the square root of a given positive number n.

Algorithm I

Step 1: Write n as $a \times b$ where a and b are close to each other.

Step 2: Then our estimate of $\sqrt{n}$ is $\frac{1}{2}(a + b)$.

The closer a and b are to each other, the better will be our estimate.

This algorithm gives us fairly good results provided we are wise in our initial choice of numbers. The following examples illustrate this.

Example 1. Let $n = 6$. Let us take $a = 2.4$ and $b = 2.5$. Then our estimate for $\sqrt{6}$ is $\frac{1}{2}(2.4 + 2.5) = 2.45$. This may be compared with the actual value which is roughly 2.4495 (an error of about 0.02%).

Example 2. Let $n = 10$. Let us take $a = 3$ and $b = 10/3$. Then our estimate for $\sqrt{10}$ is $\frac{1}{2}(3 + 10/3) = 19/6 \approx 3.167$. This may be compared with the actual value which is roughly 3.162 (an error of about 0.14%).

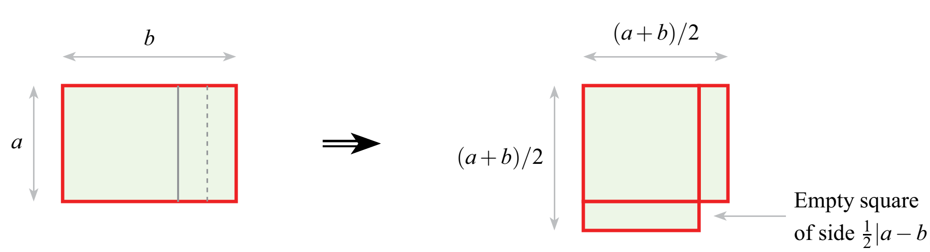

Figure 2.

We can do better that this by taking $a = 3.1$ and $b = 10/3.1$. Then our estimate for $\sqrt{10}$ is:

$$\frac{1}{2}\left(3.1 + \frac{10}{3.1}\right) = \frac{1}{2}\left(\frac{9.61 + 10}{3.1}\right) = \frac{19.61}{6.2} \approx 3.163.$$

The error now is about 0.02%.

Second approach to approximating square roots. In the study of statistics we encounter several different kinds of means defined for any given pair of positive numbers a and b. We have the **arithmetic mean** (AM), the **geometric mean** (GM) and the **harmonic mean** (HM), among many others (not listed here). These are defined as follows:

$$\text{AM} = \frac{1}{2}(a+b), \qquad \text{GM} = \sqrt{ab}, \qquad \text{HM} = \frac{2ab}{a+b}.$$

Each of these has its significance and uses. Of particular interest to us is the fact that if $a \neq b$, then:

$$\text{HM} < \text{GM} < \text{AM}.$$

These inequalities are never violated. The HM and AM *always* lie on opposite sides of the GM.

For example, if $a = 2$ and $b = 8$, then

$$\text{AM} = \frac{1}{2}(2+8) = 5, \qquad \text{GM} = \sqrt{2 \times 8} = 4,$$
$$\text{HM} = \frac{2 \times 2 \times 8}{2+8} = 3.2,$$

and, of course, $3.2 < 4 < 5$.

Recall that in our approach we start by selecting positive numbers a and b such that $n = ab$. We wish to compute the GM of a and b. But as the GM is difficult to compute, we choose to compute the AM instead. We could also choose to compute the HM (this too is an easy computation). Whichever one we choose to compute will then be our estimate for the GM.

If we use the AM, we always get an *over-estimate* for the GM. And if we use the HM, we always get an *under-estimate*.

Now an interesting idea strikes us: why not compute *both* the AM and HM, and then take their average? Their respective errors may then just cancel each other, and we may just get an estimate quite close to the GM. To our surprise, we find that this idea works very well.

So here is our *second algorithm* for approximating the square root of a given positive number n.

Algorithm II

Step 1: Write n as $a \times b$ where a and b are reasonably close to each other.

Step 2: Compute the AM and the HM of a and b.

Step 3: Compute the average of the AM and HM computed in Step 2. That is, compute:

$$\frac{1}{2}\left(\frac{a+b}{2} + \frac{2ab}{a+b}\right).$$

This is our estimate for $\sqrt{n}$.

This algorithm gives us extremely good results — far better than we would expect! The following examples illustrate this.

Example 3. Let $n = 6$. Let us take $a = 2.4$ and $b = 2.5$. Then our estimate for $\sqrt{6}$ is:

$$\frac{1}{2}\left(\frac{2.4 + 2.5}{2} + \frac{2 \times 2.4 \times 2.5}{2.4 + 2.5}\right) = \frac{1}{2}\left(2.45 + \frac{12}{4.9}\right) = \frac{4801}{1960} \approx 2.449489795.$$

Compare this with the true value: $\sqrt{6} = 2.449489742 \ldots$. The estimate is correct to seven decimal places.

Example 4. Let $n = 10$. Let us take $a = 3$ and $b = 10/3$. Then our estimate for $\sqrt{10}$ is:

$$\frac{1}{2}\left(\frac{3 + 10/3}{2} + \frac{2 \times 3 \times 10/3}{3 + 10/3}\right) = \frac{1}{2}\left(\frac{19}{6} + \frac{60}{19}\right) = \frac{721}{228} \approx 3.16228.$$

This estimate is correct to four decimal places. (The true value is 3.162277660 ....)

Example 5. Let $n = 20$. Let us take $a = 4.5 = 9/2$ and $b = 20/4.5 = 40/9$. Then our estimate for $\sqrt{20}$ is:

$$\frac{1}{2}\left(\frac{9/2 + 40/9}{2} + \frac{2 \times 9/2 \times 40/9}{9/2 + 40/9}\right) = \frac{1}{2}\left(\frac{161}{36} + \frac{720}{161}\right) = \frac{51841}{11592} \approx 4.472135955.$$

This estimate is correct to eight decimal places.

2. Estimation of Cube Roots

Let n be the given number whose cube root is required. We start by expressing n as a product abc, with a, b and c as close as possible to each other (the closer the better; here, of course, $a, b, c > 0$). The task of computing $n^{1/3} = (abc)^{1/3}$ may be expressed as: Compute the geometric mean (GM) of a, b, c. This task may be given a geometric form as follows: Construct a cube whose volume is equal to that of a cuboid with dimensions $a \times b \times c$.

As we did with the square root, we start with the arithmetic mean (AM) of a, b, c and then see how we can 'improve' it. Let $d = \frac{1}{3}(a+b+c)$ be the AM. It is well known that the AM exceeds the GM if the numbers are unequal. So let us consider subtracting a suitably small quantity h from d. We must find h so that $(d-h)^3 = n$. Write $x = d - h$, so x is going to be our estimate for the desired cube root. We argue as follows:

$$n = d^3 - 3d^2h + 3dh^2 - h^3,$$

$$\therefore\ d^3 - n \approx 3d^2h - 3dh^2$$

$$\text{(we drop the } h^3 \text{ term since } h \approx 0),$$

$$\therefore\ d^3 - n \approx 3dh(d-h),$$

$$\therefore\ \frac{d^3 - n}{3d} \approx x(d-x)$$

$$\text{(since } d - h = x \text{ and } h = d - x).$$

In the last line we change the approximation sign to an equality sign and solve the resulting equation — which is now a *quadratic equation*, not a cubic equation, and therefore easy to solve — for x. The answer we get will be our estimate for the cube root of n.

Example 6. Let us estimate the cube root of $n = 6$. Write $6 = 1 \times 2 \times 3$. Our initial estimate for the cube root of 6 is the AM of 1, 2, 3, i.e., $d = \frac{1}{3}(1+2+3) = 2$. The value of $\left(d^3 - n\right)/(3d)$ is:

$$\frac{2^3 - 6}{3 \times 2} = \frac{1}{3}$$

Hence the equation we must solve is:

$$x\,(2 - x) = \frac{1}{3}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = 1 \pm \frac{\sqrt{6}}{3}.$$

We must use the positive root. So our estimate for the cube root of 6 is:

$$1 + \frac{\sqrt{6}}{3} \approx 1.8165.$$

Here is the actual cube root, computed using a calculator:

$$6^{1/3} \approx 1.8171.$$

Our estimate is correct to two decimal places.

We can do better by writing $6 = 2 \times 2 \times 3/2$. Then we have:

$$d = \frac{1}{3}\left(2 + 2 + \frac{3}{2}\right) = \frac{11}{6},$$

$$\frac{d^3 - n}{3d} = \frac{35}{1188}.$$

Hence the equation we must solve is:

$$x\left(\frac{11}{6} - x\right) = \frac{35}{1188}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = \frac{11}{12} \pm \frac{\sqrt{127149}}{396}.$$

We must use the positive root. So our estimate for the cube root of 6 is:

$$x = \frac{11}{12} + \frac{\sqrt{127149}}{396} \approx 1.817120162.$$

This is accurate to six decimal places.

Example 7. Let us estimate the cube root of 10. Write $10 = 2 \times 2 \times 5/2$. Our initial estimate for the cube root of 10 is the AM of $2, 2, 5/2$, i.e., $d = \frac{1}{3}(2 + 2 + 5/2) = 13/6$. The value of $\left(d^3 - n\right)/(3d)$ is:

$$\frac{(13/6)^3 - 10}{13/2} = \frac{37}{1404}$$

Hence the equation we must solve is:

$$x\left(\frac{13}{6} - x\right) = \frac{37}{1404}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = \frac{13}{12} \pm \frac{\sqrt{251277}}{468}.$$

We must use the positive root. So our estimate for the cube root of 10 is:

$$\frac{13}{12} + \frac{\sqrt{251277}}{468} \approx 2.154434558.$$

Here is the actual cube root, computed using a calculator:

$$10^{1/3} \approx 2.154434690.$$

Our estimate is correct to six decimal places.

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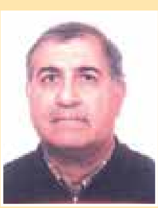

ALI IBRAHIM HUSSEN has a B.Sc. in Irrigation and Agricultural Machinery from the University of Mosul, Iraq, and a M.Sc in Civil Engg. (Hydraulic) from the University of Newcastle-Upon-Tyne, UK. He has worked for the Scientific Research Council in Baghdad, and for the Civil Engg. Dept., University of Tikrit, Iraq. He now works as a teacher of Mathematics. Ali Hussen may be contacted at Ali.Ibrahim2@kriminalvarden.se.

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All Right Then

W.H. Auden won first prize for mathematics at St. Edmund's School in Hindhead, Surrey, when he was 13. He recalled being asked to learn the following mnemonic around 1919:

Minus times Minus equals Plus;
The reason for this we need not discuss.

At Right Angles is interested in knowing how you teach this concept.
Teachers cannot get away from discussion now. So how do you explain this rule?
Do write to us at AtRiA.editor@apu.edu.in.

The best responses will be published in the next issue.