in the classroom

How to Prove It

This article continues the theme of offering multiple proofs of a single result, following entirely different themes and different starting points.

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Introduction

In the last edition of this column we studied several cases of concurrence of lines in a triangle: (i) the perpendicular bisectors of the sides, (ii) the angle bisectors, (iii) the medians. Now we study the concurrence of the *altitudes* of a triangle. What is of some interest about this story is that there are so many ways to demonstrate concurrence, and they are so different from each other. We offer four different proofs of the result.

The context is this. Let triangle ABC be given. Draw the three altitudes of the triangle, i.e., through each vertex draw the line which is perpendicular to the opposite side. The claim then is that the three altitudes meet in a point known now as the *orthocentre* of the triangle. This point may lie within the triangle or outside it. See Figure 1 for two representative pictures.

First proof: Using Pythagoras theorem

This proof deserves to be better known than it is (because it shows that we can achieve the desired result with the use of the Pythagorean theorem and nothing more), so we start here. In Figure 2 we see a $\triangle ABC$ and its altitudes AD, BE, CF. We must show that they concur. We have deliberately drawn the altitudes *incorrectly*, so they appear *not* to concur. Let BE and CF meet at K, and let L be the foot of the perpendicular from K to BC. We must show that L coincides with D, for this implies that AD passes through K, and hence that the altitudes concur.

Keywords: *Altitudes, orthocentre, concurrence, cyclic, quadrilateral, vectors*

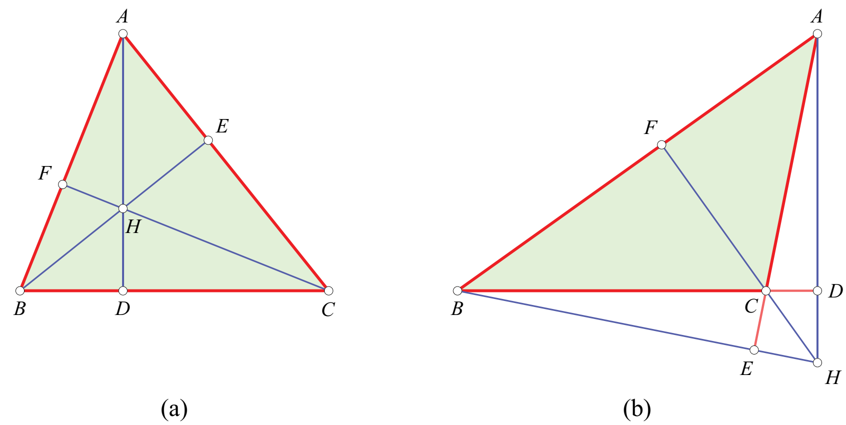

Figure 1. Concurrence of the altitudes of a triangle: Two possible configurations

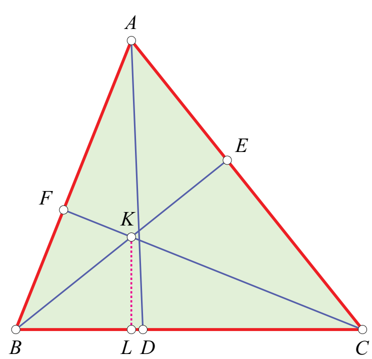

Figure 2. Proof using the Pythagorean theorem

Using the Pythagorean theorem repeatedly, we get the following relations:

$$\begin{cases} BL^2 - CL^2 &= (BL^2 + KL^2) - (CL^2 + KL^2) \\ &= KB^2 - KC^2, \\ CE^2 - AE^2 &= (CE^2 + KE^2) - (AE^2 + KE^2) \\ &= KC^2 - KA^2, \\ AF^2 - BF^2 &= (AF^2 + KF^2) - (BF^2 + KF^2) \\ &= KA^2 - KB^2. \end{cases}$$

From this it follows, by addition, that

$$BL^2 - CL^2 + CE^2 - AE^2 + AF^2 - BF^2 = 0. \quad (1)$$

We also have:

$$\begin{cases} BD^2 - CD^2 &= (BD^2 + AD^2) - (CD^2 + AD^2) \\ &= AB^2 - AC^2, \\ CE^2 - AE^2 &= (CE^2 + BE^2) - (AE^2 + BE^2) \\ &= BC^2 - BA^2, \\ AF^2 - BF^2 &= (AF^2 + CF^2) - (BF^2 + CF^2) \\ &= CA^2 - CB^2. \end{cases}$$

From this it follows, by addition, that

$$BD^2 - CD^2 + CE^2 - AE^2 + AF^2 - BF^2 = 0. \quad (2)$$

Comparing (1) and (2) we see that $BL^2 - CL^2 = BD^2 - CD^2$, and hence that

$$(BL - CL) \cdot (BL + CL) = (BD - DC) \cdot (BD + DC),$$

$$\therefore\ (BL - CL) \cdot BC = (BD - DC) \cdot BC,$$

$$\therefore\ BL - CL = BD - DC$$

(since BC is clearly not zero).

Hence $BL - BD = CL - DC$, or $LD = -LD$, giving $LD = 0$. Hence L and D are the same point. The desired end has been reached. Note that the proof has been written assuming that the points L and D lie on segment BC and not on its extension, but with small modifications (which we leave for you to do) it will hold for the other possibilities as well.

Second proof: Using circle theorems

In Figure 2 we see the altitudes BE and CF of $\triangle ABC$ intersecting at point H. The line through A and H meets line BC at D. *Note that AD is not assumed at the outset to be an altitude.* (That is why AD has been drawn in a different way from BE and CF, and the relevant angles at E and F are marked as right angles, but not the angle at D.) Rather, we must *show* that AD is an altitude. Here is how we reason it out.

- Quadrilateral $AEHF$ is cyclic because $\angle AEH$ and $\angle AHF$ are right angles. Hence $\angle AHE = \angle AFE$ ("angles in the same segment of a circle").
- Quadrilateral $BFEC$ is cyclic because $\angle BFC$ and $\angle BEC$ are right angles. Hence $\angle AFE = \angle ECB$ ("exterior angle of a cyclic quadrilateral equals the interior opposite angle").

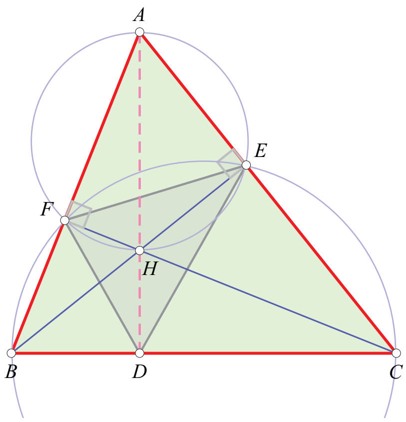

Figure 3. Proof using circle theorems

- Hence $\angle AHE = \angle ECB$. This implies that quadrilateral $DCEH$ is cyclic.
- Therefore $\angle HDC + \angle HEC = 180°$. But $\angle HEC = 90°$. Hence $\angle HDC = 90°$. That is, HD is perpendicular to BC. This the same thing as saying that $AD \perp BC$.

Third proof: Using coordinates

In adopting a coordinate-based approach, the first thing is to choose the axes wisely. In Figure 4 we see $\triangle ABC$ for which the side BC lies along the x-axis and the altitude AD lies along the y-axis. (The wisdom of this choice of axes will become apparent shortly.) Our aim will be accomplished by showing that the perpendiculars from B to line AC and from C to line AB meet on the y-axis.

Assign coordinates as shown at the right of Figure 4: $A = (0, a)$, $B = (b, 0)$, $C = (c, 0)$; here of course $b \neq c$, since B and C are distinct points. We first find the equation of the perpendicular from B to line AC. The slope of AC is $-a/c$, so the slope of the perpendicular is c/a, and the equation of the perpendicular is $y - 0 = (c/a)(x - b)$, i.e., $ay = c(x - b)$. This may be expressed as follows:

$$cx - ay = bc. \tag{3}$$

By symmetry we may deduce the equation of the perpendicular from C to line AB simply by switching the roles of b and c in the above equation. Here is what we get:

$$bx - ay = bc. \tag{4}$$

To find where the two altitudes intersect, we solve the above two equations for x and y (in fact, it is enough if we solve just for x). By subtraction we get

$$(b - c)x = 0, \quad \therefore\ x = 0. \tag{5}$$

Hence the two altitudes intersect on the y-axis, just as we wanted. It follows that the three altitudes concur.

Fourth proof: Using vectors

The last proof we offer uses vectors. Let altitudes BE and CF meet at H as in Figure 5. We wish to prove that AH is perpendicular to BC. Take H to be the origin relative to which position vectors are referred, and let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C respectively. The fact that

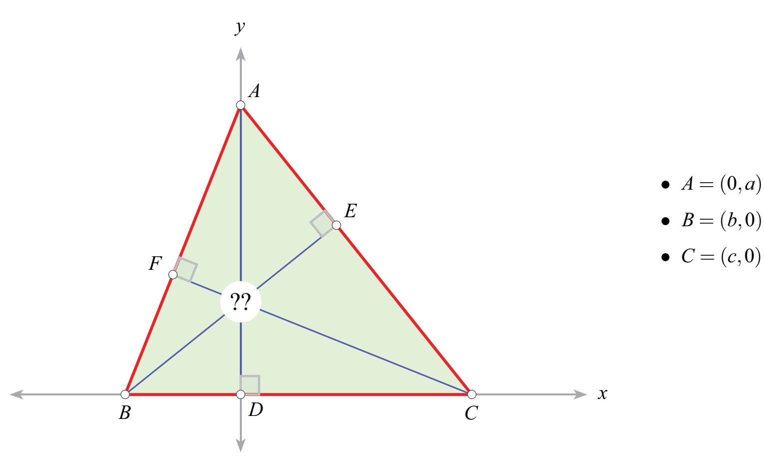

Figure 4. Proof using coordinates. The '??' indicates that we do not know whether or not the lines concur.

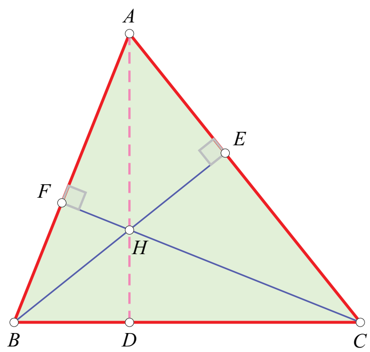

Figure 5. Last but not least: a vector proof

$BH \perp AC$ and $CH \perp AB$ then expresses itself in the following statements:

$$\begin{cases} \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0, \\ \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0. \end{cases} \tag{6}$$

The first of these yields

$$\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b},$$

while the second one yields

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}.$$

Hence we have: $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, that is:

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0.$$

This is the same thing as saying that $AH \perp BC$.

Here is a query for you to ponder: Is the vector proof really that different from the proof based on coordinates?

Closing comments

So there you have it: four entirely different proofs for a single result. You may find it instructive to compare them and contrast their distinctive features.

Does this compilation exhaust the list of proofs available for this result? Not at all! We can think of at least two more proofs. One proceeds by constructing a triangle with twice the dimensions of the given triangle and invoking the midpoint theorem and the properties of a parallelogram; this may well be the most elegant proof available. Yet another proof uses the idea of the *radical axis* of two circles. We leave you to explore these for yourself.

In closing we point out the following interesting and attractive result. For any triangle ABC, let H denote its orthocentre. Consider the set of vertices $\{A, B, C, H\}$. This set has the following property: *For each subset of three points from the set, form a triangle with those points as its vertices; then the fourth point is the orthocentre of this triangle.* In other words, if H is the orthocentre of $\triangle ABC$, then A is the orthocentre of $\triangle HBC$; B is the orthocentre of $\triangle HCA$; and C is the orthocentre of $\triangle HAB$. For this reason, the set $\{A, B, C, H\}$ is called an *orthocentric set.*

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