3, 4, 5 ...

And other memorable triples — Part II

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feature

In Part I of this article we had showcased the triple $(3, 4, 5)$ by highlighting some of its properties and some configurations where it occurred naturally. We now attempt to extend this to other triples of consecutive integers. To begin with, we study the two 'siblings' of $(3, 4, 5)$, namely, the triples $(2, 3, 4)$ and $(4, 5, 6)$. We start first with the elder sibling, $(4, 5, 6)$. (We do need to show the older ones some respect, don't we?)

The triple 4, 5, 6

In Figure 1 we see a sketch of a triangle ABC with sides 4, 5, 6 (with $a = 6, b = 5, c = 4$). Is there anything special about the triangle? Let's do some exploration using *GeoGebra*.

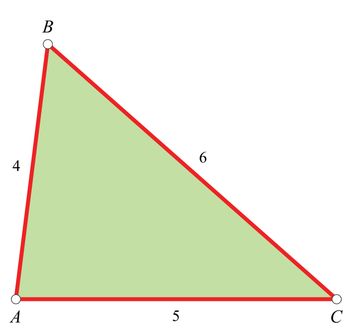


Figure 1.

Keywords: *Triangle, consecutive integers, triple, double angle, sine rule, cosine rule, Pythagoras*

Figure 1 shows a *GeoGebra* sketch of the triangle. We start by measuring the angles of the triangle (using the tool available in *GeoGebra*). Here is the output:

$$\measuredangle A = 82.82°, \qquad \measuredangle B = 55.77°, \qquad \measuredangle C = 41.41°.$$

Examining the data, we quickly notice that 82.82 is twice 41.41, in other words: $\measuredangle A = 2\measuredangle C$. Right away we have uncovered something notable and of interest!

But wait: this relation has been *numerically determined*. Could it be the case that if we compute both angle measures to more decimal places than shown above, the above relation will turn out to be only approximate and not exact? How can we check whether or not $\measuredangle A$ is *exactly* twice $\measuredangle C$?

We can do so using trigonometry. Let us compute the cosines of all three angles of the triangle using the cosine rule:

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{25+16-36}{2\times 20} = \frac{1}{8},$$
$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{16+36-25}{2\times 24} = \frac{9}{16},$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{36+25-16}{2\times 30} = \frac{3}{4}.$$

To see if $\measuredangle A = 2\measuredangle C$ as suggested by the empirical evidence, we must check whether $\cos A = 2\cos^2 C - 1$ (for we have the identity $\cos 2\theta = 2\cos^2\theta - 1$ which is true for any angle θ). We have:

$$2\cos^2 C - 1 = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{9}{8} - 1 = \frac{1}{8} = \cos A,$$

and since both $\measuredangle A$ and $\measuredangle C$ are acute angles, the verification is complete. *So the relation $\measuredangle A = 2\measuredangle C$ is indeed exact.*

The same property can be proved by a geometric argument which may be preferred by some. In Figure 2 (a) we have redrawn the 4-5-6 triangle with the perpendicular BD from vertex B to side AC. Our first task is to find the length x of AD. We shall make use of the Pythagorean theorem to do so. Let h be the length of BD. Then we have:

$$h^2 + x^2 = 4^2,$$
$$h^2 + (5-x)^2 = 6^2,$$

hence by subtraction: $(5-x)^2 - x^2 = 6^2 - 4^2$, i.e., $25 - 10x = 20$. This yields $x = 1/2$.

Let E be the point on side AC such that $AE = 1$ unit; see Figure 2 (b). Join BE. Since $DE = DA$, it follows that $BE = BA$. Also $EC = 5 - 1 = 4$ units. So we have $AB = BE = EC$. Hence $\measuredangle BEA = 2\measuredangle BCA$, and also $\measuredangle BEA = \measuredangle BAE$. It follows that $\measuredangle BAC = 2\measuredangle BCA$, i.e., $\measuredangle A = 2\measuredangle C$.

A Stronger Property

We now prove something much more striking:

Theorem 1. *There is only one triple of consecutive integers with the property that the triangle with these numbers as its side lengths has one angle which is twice another one. This is the triple* (4, 5, 6).

Let the sides of the triangle be $n, n+1, n+2$. Let the triangle be labelled ABC so that $a = n+2$, $b = n+1$, $c = n$. Since $a > b > c$, we have $\measuredangle A > \measuredangle B > \measuredangle C$. So if one angle of the triangle is twice another, one of the following must be true: (i) $\measuredangle A = 2\measuredangle B$ (ii) $\measuredangle B = 2\measuredangle C$ (iii) $\measuredangle A = 2\measuredangle C$.

There are now two ways of proceeding. One is to use the cosine rule. This works, but the algebra is

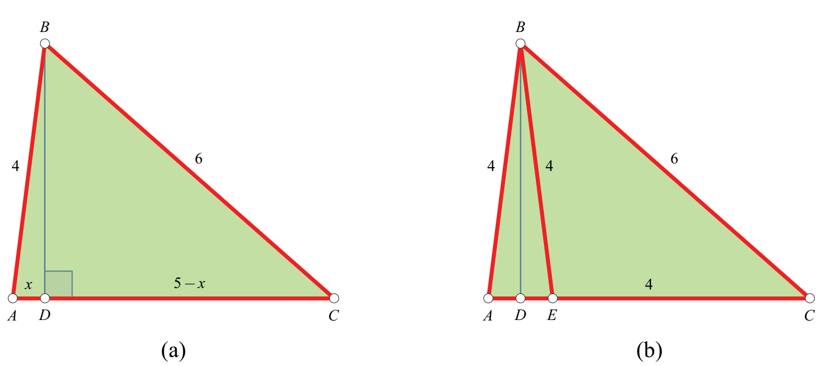


Figure 2.

messy. The other, which is more interesting as well as more efficient, and which we prefer, is to use a geometric Pythagoras-style theorem which is striking by itself.

Theorem 2. *Let $\triangle ABC$ have sides a, b, c. Then the relation $\measuredangle A = 2\measuredangle B$ is true if and only if $a^2 = b(b+c)$.*

Proof of Theorem 2: Forward implication. We first tackle the statement: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. (This is the 'only if' part of the theorem.) We offer a trigonometric proof of the result. Let $\measuredangle B = \theta$; then $\measuredangle A = 2\theta$ and $\measuredangle C = 180^\circ - 3\theta$. Hence we have $\sin A = \sin 2\theta$ and $\sin C = \sin 3\theta$. The sine rule yields:

$$\frac{a}{\sin 2\theta} = \frac{b}{\sin\theta} = \frac{c}{\sin 3\theta}.$$

From the first equality we get:

$$a = b\cdot\frac{\sin 2\theta}{\sin\theta} = 2b\cos\theta, \qquad \therefore\ \cos\theta = \frac{a}{2b}.$$

The second equality yields:

$$\begin{aligned} c = b\cdot\frac{\sin 3\theta}{\sin\theta} &= b\cdot\frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} \\ &= b\left(3 - 4\sin^2\theta\right) \\ &= b\left(4\cos^2\theta - 1\right). \end{aligned}$$

Substituting for $\cos\theta$ in this relation, we get:

$$c = b\left(\frac{a^2}{b^2} - 1\right) = \frac{a^2 - b^2}{b},$$
$$\therefore\ a^2 = b^2 + bc = b(b+c),$$

as claimed.

Proof of Theorem 2: Reverse implication. Now we tackle the 'if' part of the theorem, namely: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. Once again, we offer a trigonometric proof of the result. We use the sine rule together with the following beautiful identity whose proof we leave as an exercise:

$$\sin^2 A - \sin^2 B = \sin(A+B)\ \sin(A-B).$$

The sine rule tells us that for any triangle ABC, we have $a/\sin A = b/\sin B = c/\sin C =$ some constant k. (In fact, k is the circum-diameter of the triangle, i.e., it is twice the radius of the circumcircle. But we do not need this information right now.)

From the relation $a^2 = b(b+c)$ we get $a^2 - b^2 = bc$, which tells us that $a > b$ and therefore that $\measuredangle A > \measuredangle B$. The same relation also yields, by the sine rule:

$$\sin^2 A - \sin^2 B = \sin B\ \sin C.$$

Using the trigonometric identity quoted above, we get:

$$\sin(A+B)\ \sin(A-B) = \sin B\ \sin C.$$

Since $A + B + C = 180^\circ$, we have $\sin(A+B) = \sin C$. Since $\sin C \neq 0$, we get:

$$\sin(A-B) = \sin B.$$

Since $A - B$ and B lie between 0° and 180° and have equal sine, they are either equal angles or they are supplementary angles. The latter possibility leads to $(A-B)+B = 180^\circ$, i.e., $A = 180^\circ$, which is absurd. Hence this case does not hold. It follows that $A - B = B$, i.e., $\measuredangle A = 2\measuredangle B$.

There is also an elegant geometric proof of the result (both parts: forward implication as well as reverse implication), which we shall discuss later.

Proof of Theorem 1. We now use Theorem 2 to prove Theorem 1. We consider the three possibilities in turn.

Case (i): If $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$, hence:

$$\begin{aligned} (n+2)^2 &= (n+1)(2n+1), \\ \therefore\ n^2 + 4n + 4 &= 2n^2 + 3n + 1, \\ \therefore\ n^2 - n - 3 &= 0. \end{aligned}$$

This equation has roots $n = \frac{1}{2}(1 \pm \sqrt{13})$. These are not positive integers (or even rational numbers), so we do not get any solution from this possibility.

Case (ii): If $\measuredangle B = 2\measuredangle C$, then $b^2 = c(c+a)$, hence:

$$\begin{aligned} (n+1)^2 &= n(2n+2), \\ \therefore\ (n-1)(n+1) &= 0. \end{aligned}$$

This yields $n = \pm 1$. Only the positive sign is of interest to us. However, the triangle corresponding to $n = 1$ has sides $1, 2, 3$ and so is degenerate: it is 'flat', with angles 180°, 0° and 0°. Note that the solution is not 'wrong'. For, this triangle has $\measuredangle B = 0^\circ = \measuredangle C$, which means that we do have the relation $\measuredangle B = 2\measuredangle C$! But it is of no interest to us, so we move on.

Case (iii): If $\measuredangle A = 2\measuredangle C$, then $a^2 = c(c+b)$, hence:

$$(n+2)^2 = n(2n+1),$$
$$\therefore\ n^2 + 4n + 4 = 2n^2 + n,$$
$$\therefore\ n^2 - 3n - 4 = 0,$$
$$\therefore\ (n+1)(n-4) = 0.$$

The last equation has roots $n = -1$ and $n = 4$. We finally do get a positive integral root, $n = 4$, and this yields a genuine, well-behaved triangle: a triangle with sides $4, 5, 6$. This yields a solution to the stated problem.

It follows that there is precisely one triangle with the stated property: the one that has sides $4, 5, 6$.

In closing we may say that the triple $(4, 5, 6)$ can lay its own claim to fame, with its own pleasing property, just like its better known sibling $(3, 4, 5)$.

A Geometric Proof of Theorem 2

Some readers may prefer to see a *geometric* proof of Theorem 2 (we had earlier given a proof using trigonometry). We offer one such proof here.

First we deal with the forward implication: if $\measuredangle A = 2\measuredangle B$, then $a^2 = b(b+c)$. The relevant configuration is shown in Figure 3.

We need an auxiliary construction. Draw a circle tangent to side BC at B and passing through vertex A. (The circle may be constructed as follows: draw a perpendicular to BC through B, and draw the perpendicular bisector of side AB; the point where these two lines meet is then the centre of the desired circle. These auxiliary construction lines have not been shown in Figure 3, to avoid a visual clutter.)

Extend side CA beyond vertex A to meet the circle again at point D. Draw segments BD and AD, as shown. Let AD have length d. Let $\measuredangle ABC = \theta$; then $\measuredangle BAC = 2\theta$ as per the given data.

From the fact that CB is tangent to the circle at B, two deductions follow: (i) $\measuredangle ABC = \measuredangle ADB$, i.e., $\measuredangle ADB = \theta$; this follows from the "angle in the alternate segment" theorem; (ii) $CB^2 = CA \times CD$, i.e., $a^2 = b(b+d)$; this is true because CAD is a secant.

Since $\measuredangle BAC = \measuredangle ADB + \measuredangle ABD$, it follows that $\measuredangle ADB = \theta$. Hence $\triangle ADB$ is isosceles, with $AD = AB$. So $d = c$. Combining this with deduction (ii), above, we see that $a^2 = b(b+c)$, as claimed.

Now for the reverse implication: if $a^2 = b(b+c)$, then $\measuredangle A = 2\measuredangle B$. We use the same figure for the proof, with the same auxiliary construction. The configuration is depicted in Figure 4. As earlier, we have drawn a circle tangent to side BC at B and passing through vertex A; then we have extended side CA beyond vertex A to meet the circle again at point D, and drawn segments BD and AD. Let AD have length d.

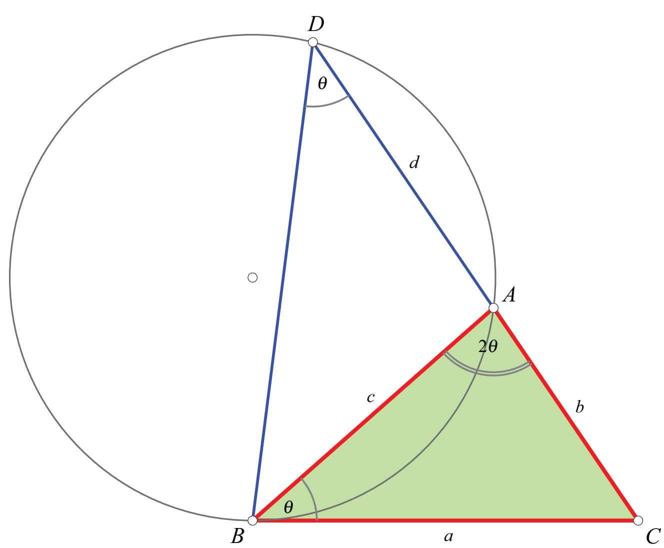


Figure 3. Given that ∡A = 2∡B, to show that $a^2 = b(b+c)$

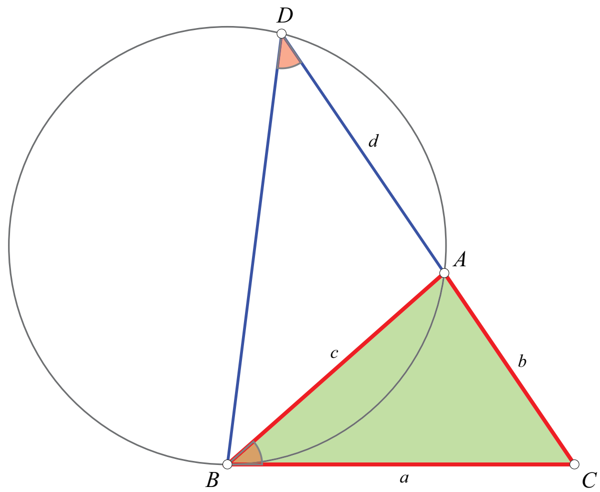


Figure 4. Given that $a^2 = b(b + c)$, to show that $\measuredangle A = 2\measuredangle B$

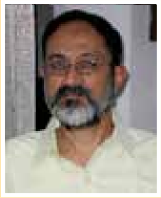
Since CB is tangent to the circle at B, and CAD is a secant, we have the following relation: $CB^2 = CA \times CD$, i.e., $a^2 = b(b + d)$.

But we also have the given relation $a^2 = b(b + c)$. Comparing the two relations, we conclude that $c = d$, i.e., $AB = AD$. Hence $\measuredangle ABD = \measuredangle ADB$. And since $\measuredangle BAC = \measuredangle ABD + \measuredangle ADB$, it follows that $\measuredangle BAC = 2\measuredangle ADB$.

But we also have $\measuredangle ABC = \measuredangle ADB$, by the "angle in the alternate segment" theorem. Hence $\measuredangle BAC = 2\measuredangle ABC$, i.e., $\measuredangle A = 2\measuredangle B$, as claimed.

Appendix: Integer triples associated with this theorem

Associated with the Pythagorean theorem we have the number theoretic problem of generating Pythagorean triples. In the same way, associated with the main result derived in this article, we have another interesting number theoretic problem: that of generating integer triples (a, b, c) which satisfy the equation $a^2 = b(b + c)$. We may want to impose the additional condition that a, b, c are coprime, just as we did in the case of Pythagorean triples. We already have one example of such a triple: $(6, 4, 5)$. Are there any others? Yes; and they are quite easy to find. We leave this question for the reader to tackle: that of finding an efficient and effective algorithm for generating all coprime integer triples (a, b, c) which satisfy the equation $a^2 = b(b + c)$. We will take up a study of this equation in a subsequent article.



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