An impossible relation

In the accompanying article on *Tangrams*, a claim was made that it is not possible to find integers a and b which make any of the following equalities true:

$$\sqrt{6} = a + b\sqrt{2}, \qquad \sqrt{7} = a + b\sqrt{2}, \qquad \sqrt{12} = a + b\sqrt{2},$$

and so on. However, the proofs may not be obvious. In this brief note, we shall prove the impossibility of the first relation and leave the remaining ones for you to handle.

Claim. *It is not possible to find rational numbers a and b such that $\sqrt{6} = a + b\sqrt{2}$.* (Note that we have replaced the word 'integers' by 'rational numbers.' Thus we are proving a stronger version of the statement than the original one.)

Proof. As with the proofs of most assertions of this kind, this is a proof by contradiction. We shall assume that there do exist rational numbers a and b for which $\sqrt{6} = a + b\sqrt{2}$ and then show that this assumption leads to a contradiction.

As readers must be familiar with the well-known proof of the irrationality of $\sqrt{2}$, we shall not bother with repeating the proof. By using virtually the same reasoning, we can also prove the irrationality of the following numbers: $\sqrt{3}$ and $\sqrt{6}$. We shall assume that you have already gone through this exercise.

In the relation $\sqrt{6} = a + b\sqrt{2}$, it cannot be that $b = 0$, for this would mean that $\sqrt{6}$ is a rational number. Hence $b \neq 0$. It also cannot be that $a = 0$. For, if $a = 0$, then by division we would get $\sqrt{3} = b$, which would mean that $\sqrt{3}$ is a rational number. However, we know that this is not true. Hence $a \neq 0$. So both a and b are non-zero.

Squaring both sides of the relation $\sqrt{6} = a + b\sqrt{2}$, we get: $6 = a^2 + 2b^2 + 2ab\sqrt{2}$, hence:

$$\sqrt{2} = \frac{6 - a^2 - 2b^2}{2ab}.$$

In this relation, the denominator is non-zero, implying that $\sqrt{2}$ is a rational number. However, we know that this is not true. Hence the stated relation cannot hold. That is, it is not possible to find rational numbers a and b such that $\sqrt{6} = a + b\sqrt{2}$. □

— $\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$