Theorems on Magic Squares

Shailesh Shirali

As has been pointed out in the companion piece on the same topic elsewhere in this issue, magic squares have been a source of recreation and leisure from ancient times. There is something about the symmetry and patterns contained in such squares that carry great appeal. In this piece, we shall prove two simple results about 3×3 and 4×4 magic squares. (Such squares would also be called *third-order* and *fourth-order* magic squares respectively.) Considering the ease and elegance with which they can be proved, the results will only add further appeal to an already wonderful subject.

Terminology. A magic square of order n is an $n \times n$ array of distinct positive integers with the property that its rows, its columns and its two main diagonals all have the same sum. This sum is called the ***magic sum*** of the square, or the *magic constant* of the square. A ***line*** of a magic square is any row, any column or either of the two main diagonals of the square. A magic square of order n thus has $2n + 2$ such lines.

Structure of a Third-Order Magic Square

We shall prove the following simple and pleasing properties which are exhibited by any third-order magic square. Let s be the magic sum of such a square, and let m be the number in the central cell of the square (i.e., the number in row # 2 and column # 2).

Keywords: *Magic square, order, magic sum, line, arithmetic progression, symmetry*

Then we have:

- $s = 3m$;
- There are four distinct three-term arithmetic progressions (APs) within the square: (i) the numbers in the central row, (ii) the numbers in the central column, (iii) the numbers in each of the two diagonals.

Note that if we prove the first assertion, the second one gets proved as well. For, if the three numbers in any of the triples referred to above are a, m, b, then we have $a + m + b = s = 3m$ (by the first assertion), hence $a + b = 2m$, i.e., $m - a = b - m$. This proves that a, m, b form an AP.

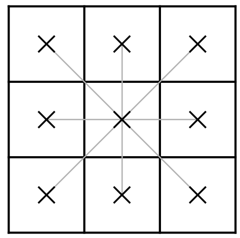


Figure 1

To prove the first assertion ($s = 3m$), we consider the four lines going through the central square (see Figure 1). The sum of the three numbers on each line is s, hence the four lines together yield a sum of $4s$, with some numbers getting counted more than once. Now note that the lines pass through every cell in the array, but the central square (which lies on all the lines) gets 'covered' four times. So the lines cover every cell in the whole array, with the central square getting covered an 'extra' three times. The sum of all the numbers in the whole array is equal to $3s$. These facts together lead to the following equation:

$$4s = 3s + 3m.$$

Hence $s = 3m$, as claimed.

Here are some notable consequences of this result: if the numbers in a third-order magic square are the numbers from 1 to 9, then the number in the central cell is necessarily 5. For, the sum of the numbers from 1 to 9 is 45, so the magic sum is $s = 45/3 = 15$. This yields $m = 5$.

Next, using the numbers from 1 to 9, the only three-term arithmetic progressions with central term 5 and sum 15 are {1, 5, 9}, {2, 5, 8}, {3, 5, 7} and {4, 5, 6}. These four APs correspond (in some order) to the four lines that pass through the central cell of the square. Now consider the lines that contain 1. Since the total of the three numbers in any line is 15, the sum of the other two numbers in such a line must be 14. The pairs which yield a sum of 14 are the following: {5, 9}, {6, 8}. Observe that there are only two such pairs. This implies that 1 cannot occur in a corner of the array (for that would require three pairs). Hence 1 occurs in the middle of a border row or column. By applying a suitable rotation, we can bring the 1 to the top row; this will naturally not disturb the 'magic' property. The two lines of which 1 is a part must have the pairs {5, 9} and {6, 8} for the remaining two numbers. Of these, the former pair corresponds to the line going through the centre of the square. The 6 and 8 may be filled in the top corner cells in either order. Once this is done, the remaining cells get filled on their own. The result is shown in Figure 2.

8	1	6
3	5	7
4	9	2

Figure 2

Structure of a Fourth-Order Magic Square

Fourth-order magic squares have rather more complex symmetries than third-order magic squares. One such symmetry is indicated in Figure 3. We shall prove that if p, q, r, s are the numbers in the cells as indicated, then the following equality necessarily holds:

$$p + q = r + s.$$

p			q
	r	s	

Figure 3

To prove the claim, it is convenient to use the symbols shown in the array below.

a_1	b_1	c_1	d_1
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
a_4	b_4	c_4	d_4

We must prove that $a_1 + d_1 = b_4 + c_4$. We use repeatedly the defining property of a magic square. The sum of the numbers in the two main diagonals equals the sum of the numbers in the two middle columns, hence:

$$(a_1 + b_2 + c_3 + d_4) + (a_4 + b_3 + c_2 + d_1) = (b_1 + b_2 + b_3 + b_4)+(c_1 + c_2 + c_3 + c_4)\ .$$

On cancellation of common terms, this simplifies to:

$$a_1 + d_4 + a_4 + d_1 = b_1 + b_4 + c_1 + c_4.$$

We rewrite this as follows:

(1) $$a_1 + d_1 - b_1 - c_1 = b_4 + c_4 - a_4 - d_4.$$

We also have:

(2) $$a_1 + b_1 + c_1 + d_1 = a_4 + b_4 + c_4 + d_4.$$

Adding equations (1) and (2) we get $a_1 + d_1 = b_4 + c_4$.

Having proved this relation, a number of further such symmetries can be anticipated. Thus, the relation $p + q = r + s$ will hold in each of the diagrams shown in Figure 4. The method of proof will be identical in all three cases.

	p	q	
r			s

	r	s	
p			q

p			
		s	
	r		
			q

Figure 4

Custom-made fourth-order magic squares! The relations uncovered above provide us with a way for constructing fourth-order magic squares in which the numbers in the top row have been filled in an arbitrary manner. For example, they could be the numbers which give your birth date. To show how this is done, we construct a magic square in which the top row has the numbers 15, 8, 19, 47 (these numbers codify an important date in Indian history).

15	8	19	47
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
a_4	b_4	c_4	d_4

The magic sum of the square is 15 + 8 + 19 + 47 = 89. There are as many as twelve unknowns in this array! The first question is: where do we start? We choose to start with the corner cells of the bottom row. We have:

$$a_4 + d_4 = 8 + 19 = 27.$$

Let us arbitrarily assign a pair of values to a_4, d_4, keeping the above condition in mind. Of course, we make sure that we do not use any of the numbers that have already occurred in the top row. Let us choose: $a_4 = 10$, $d_4 = 17$. (The choice is purely arbitrary.) We now update the array:

15	8	19	47
a_2	b_2	c_2	d_2
a_3	b_3	c_3	d_3
10	b_4	c_4	17

Next we choose to fill the cells in the central 2 × 2 block. We have:

$$b_2 + c_3 = 10 + 47 = 57,$$

$$b_3 + c_2 = 15 + 17 = 32.$$

We arbitrarily select: $b_2 = 5$, $c_3 = 52$, $b_3 = 28$, $c_2 = 4$. These yield: $b_4 = 89 - (8 + 5 + 28) = 48$, $c_4 = 89 - (19 + 4 + 52) = 14$. Once again, our choices must be such that no numbers are repeated. We now update the array:

15	8	19	47
a_2	5	4	d_2
a_3	28	52	d_3
10	48	14	17

Now write x for a_2. Then we have, since the magic sum is 89:

$$d_2 = 89 - 9 - x = 80 - x, a_3 = 89 - 25 - x = 64 - x,$$

$$d_3 = 89 - 64 - (80 - x) = x - 55.$$

As all the numbers are required to be positive, we must have the following:

$$x > 55, x < 64, \quad \therefore 56 \le x \le 63.$$

The numbers must also all be unequal, hence x, $80 - x$, $64 - x$, $x - 55$ must be different from all of the following:

15, 8, 19, 47, 5, 4, 28, 52, 10, 17, 48, 14.

Trying out the choices one by one, we find that $x = 57$ works. Here is the magic square it yields:

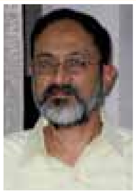
15	8	19	47
57	5	4	23
7	28	52	2
10	48	14	17

The reader may remark here that we have been lucky: no cases occurred of repeated numbers. Yes, we were lucky. But if indeed some repetition of numbers had happened, all that we would have to do is backtrack and make a different choice at some stage. In general, there are enough numbers available that we will obtain what we seek!

Exercises

Construct fourth-order magic squares in which the numbers in the first row are as given below:

1. (22, 12, 18, 87); this refers to Ramanujan's birthday (22^{nd} December, 1887);
2. (14, 3, 18, 79); this refers to Albert Einstein's birthday (14^{th} March, 1879);
3. Your own birthday!



SHAILESH SHIRALI is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.