in the classroom

How To Prove It

In this article, we offer a second proof of the triangle-in-a-triangle theorem, using the principles of similarity geometry. Then, using vectors, we prove a result which is a generalisation of that theorem. We also give a pure geometry proof of the generalisation.

SHAILESH A SHIRALI

Given an arbitrary $\triangle ABC$ and a number t between 0 and 1, we locate points D, E, F on sides BC, CA, AB (see Figure 1) such that

$$\frac{BD}{BC} = \frac{CE}{CA} = \frac{AF}{AB} = t.$$

Segments AD, BE, CF when drawn intersect and demarcate a triangle PQR within the larger triangle ABC. The question now is: What is the ratio $f(t)$ of the area of $\triangle PQR$ to that of $\triangle ABC$? We showed in the previous issue that

$$f(t) = \frac{(2t-1)^2}{1-t+t^2}. \tag{1}$$

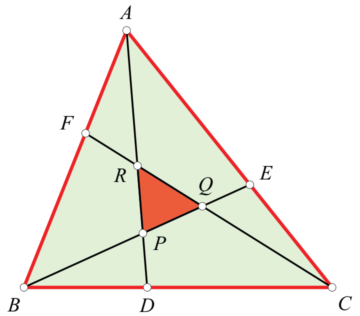


Figure 1.

Keywords: *Routh's theorem, triangle, ratio, area, similarity, vector*

In this issue, we shall modify the proof to obtain a generalisation of this result, known as *Routh's theorem*. But before we do that, we offer a 'pure geometry' proof for the formula for $f(t)$. We show that the derivation can be done by using the geometry of similar triangles. The derivation is due to **Swati Sircar** of Azim Premji University.

A Pure Geometry Proof

In general, a solution by pure geometry involves the construction of a few (appropriately and ingeniously chosen) auxiliary lines and circles. Here the steps we perform are the following: draw lines through B, C, A parallel respectively to segments AD, BE, CF. These lines intersect in pairs and create triangle XYZ (Figure 2). Extend segments AD, BE, CF to meet lines XY, YZ, ZX at points U, V, W respectively. Observe that in the resulting figure there are numerous triangles similar to $\triangle PQR$. We shall use these similarity relations to arrive at the desired answer.

Our strategy will be to first find the ratios $AR : RP : PU$, $BP : PQ : QV$ and $CQ : QR : RW$. We start by examining the ratios $BP : PQ : QV$. We already know the ratio $BQ : QV$, for by similarity,

$$\frac{BQ}{QV} = \frac{BF}{FA} = \frac{1-t}{t}.$$

As the ratio $(1-t)/t$ recurs all through the computation, it is convenient to have a symbol to denote it. Let $k = (1-t)/t$; then $BF/FA = AE/EC = CD/DB = k$, and also $BQ/QV = k$, $AP/PU = k$, $CR/RW = k$.

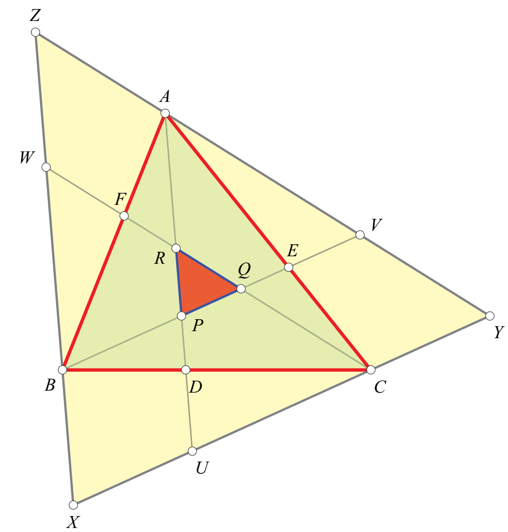


Figure 2.

Let $PQ/QV = \alpha$; we must determine α. We then have (the situation being schematically depicted as in Figure 3):

$$BP : PQ : QV = (k-\alpha) : \alpha : 1. \qquad (2)$$

By similarity we have $AR : RP = VQ : QP$. Since $AP : PU = k$, it follows that

$$AR : RP : PU = k : k\alpha : (\alpha + 1). \qquad (3)$$

Next, we have $CR : RW = k$ and

$$\frac{CQ}{QR} = \frac{UP}{PR} = \frac{1+\alpha}{k\alpha}.$$

Hence $CQ : RW = k : 1 + \alpha$, and:

$$CQ : QR : RW = k\big(1 + (k+1)\alpha\big) : k^2\alpha : 1 + (k+1)\alpha. \qquad (4)$$

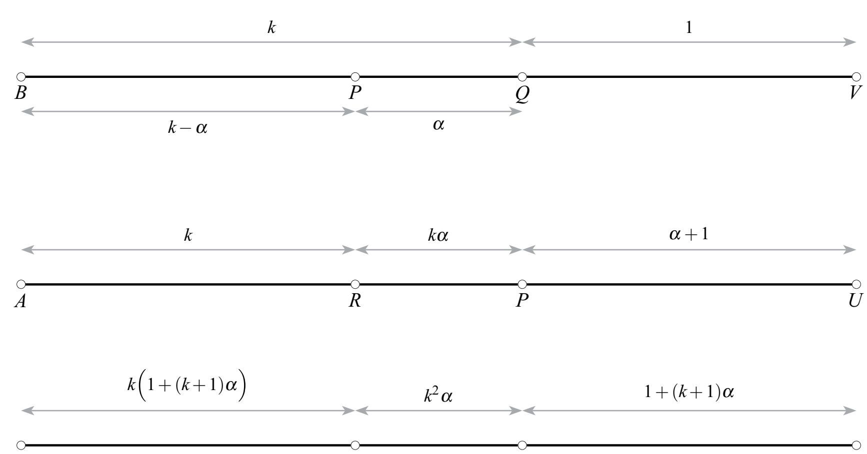


Figure 3.

Now we return to the original segment $BPQV$ (so the wheel has "turned full circle"). Since $BP : PQ = WR : RQ$, we have:

$$\frac{1+(k+1)\alpha}{k^2\alpha} = \frac{k-\alpha}{\alpha},$$

$$\therefore\ \alpha + (k+1)\alpha^2 = k^3\alpha - k^2\alpha^2,$$

$$\therefore\ \alpha = \frac{k^3-1}{k^2+k+1} = k-1,$$

a compact and pleasing result. On substituting this into the various expressions, we find that

$$BP : PQ : QV = AR : RP : PU = CQ : QR \\ : RW = 1 : k-1 : 1 = t : 1-2t : t. \qquad (5)$$

In particular we have $BP = QV$, $AR = PU$ and $CQ = RW$.

Next we determine the ratio $QE : QV$. We have:

$$\frac{QE}{EV} = \frac{EC}{AE} = \frac{t}{1-t}, \qquad \therefore\ \frac{QE}{QV} = t.$$

This implies that

$$BP : PQ : QE = t : 1-2t : t^2, \qquad (6)$$

and it yields:

$$\frac{BQ}{BE} = \frac{1-t}{1-t+t^2}. \qquad (7)$$

The ratios $AP : AD$ and $CR : CF$ are given by the same expression. The rest of the derivation proceeds as in the vector solution: the ratio of the sum of areas of $\triangle ABP$, $\triangle BCQ$ and $\triangle CAR$ to $\triangle ABC$ is

$$\frac{1-t}{1-t+t^2} \times 3t = \frac{3t(1-t)}{1-t+t^2},$$

hence the required ratio is 1 minus this quantity, i.e.:

$$\frac{\text{Area}\ (\triangle PQR)}{\text{Area}\ (\triangle ABC)} = 1 - \frac{3t(1-t)}{1-t+t^2} \\ = \frac{(1-2t)^2}{1-t+t^2}. \qquad (8)$$

Remark. Before proceeding, we pause to consider a special case of the above formula. Put $t = n/(2n+1)$; then we get, after simplification:

$$\frac{\text{Area}\ (\triangle PQR)}{\text{Area}\ (\triangle ABC)} = \frac{1}{1+3n+3n^2}.$$

This value of t corresponds to dividing the sides of the triangle into $2n+1$ equal parts (using $2n$ points equally spaced along the sides) and requiring that D, E, F lie at the points of division which are closest to the respective side midpoints. As you can see, it results in a simple formula for the areal ratio. The choice $2n+1 = 3$ (which comes from $n = 1$) corresponds to trisecting the sides; it results in the areal ratio $1 : 7$.

The denominator in the above formula, $3n^2 + 3n + 1$, generates the following sequence (by putting $n = 0, 1, 2, \ldots$):

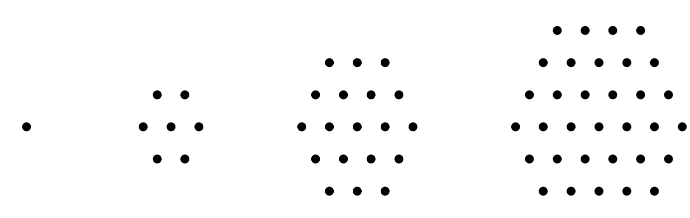
$$1,\ 7,\ 19,\ 37,\ 61,\ 91,\ \ldots.$$

These numbers are the differences between consecutive cubes: $1 = 1^3 - 0^3$, $7 = 2^3 - 1^3$, $19 = 3^3 - 2^3$, and so on. To see why this is true in general, note the following simple identity: $3n^2 + 3n + 1 = (n+1)^3 - n^3$. The very same numbers are generated by the sequence of centred hexagonal dot figures shown in Figure 4. This explains why they are sometimes called the *centred hexagonal numbers*.

Routh's Theorem

Now we consider a more somewhat general configuration. We start as earlier with an arbitrary $\triangle ABC$, but now we have three numbers u, v, w (rather than just one number t), all between 0 and 1. We now locate points D, E, F on sides BC, CA, AB (see Figure 5) such that

$$\frac{BD}{BC} = u, \quad \frac{CE}{CA} = v, \quad \frac{AF}{AB} = w.$$



1　　7　　19　　37

Figure 4.

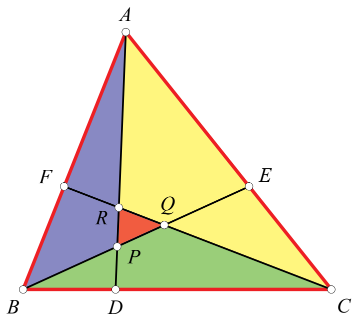


Figure 5.

As earlier, segments AD, BE, CF when drawn intersect and demarcate a triangle PQR within the larger triangle ABC. The question now is: What is the ratio $g(u,v,w)$ of the area of $\triangle PQR$ to that of $\triangle ABC$?

Let B be treated as the origin, and let

$$\overrightarrow{BC} = \mathbf{c}, \qquad \overrightarrow{BA} = \mathbf{a}.$$

By construction we have

$$\overrightarrow{BD} = u\,\overrightarrow{BC} = u\mathbf{c}, \quad \overrightarrow{CE} = v\,\overrightarrow{CA} = v\,(\mathbf{a} - \mathbf{c}),$$
$$\overrightarrow{AF} = w\,\overrightarrow{AB} = -w\mathbf{a}.$$

Let $AP/AD = k$. To find the unknown quantity k, we argue as follows.

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = -\mathbf{a} + u\mathbf{c},$$
$$\therefore\ \overrightarrow{AP} = k\,\overrightarrow{AD} = -k\mathbf{a} + ku\mathbf{c},$$
$$\therefore\ \overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP} = (1-k)\mathbf{a} + ku\mathbf{c}.$$

We also have, by similar logic or by using the section formula:

$$\overrightarrow{BE} = v\mathbf{a} + (1-v)\mathbf{c}.$$

Now consider the last two results we have obtained:

$$\overrightarrow{BP} = (1-k)\mathbf{a} + ku\mathbf{c}, \tag{9}$$
$$\overrightarrow{BE} = v\mathbf{a} + (1-v)\mathbf{c}. \tag{10}$$

Vectors $\overrightarrow{BP}$ and $\overrightarrow{BE}$ are parallel and have been expressed in terms of the non-zero, non-parallel vectors **a** and **c**. Hence **a** and **c** must be mixed in the same proportions in $\overrightarrow{BP}$ and $\overrightarrow{BE}$, and we have:

$$\frac{1-k}{ku} = \frac{v}{1-v}. \tag{11}$$

This allows us to find the unknown quantity k:

$$\frac{AP}{AD} = \frac{1-v}{1-v+uv}. \tag{12}$$

In the same way (or by using symmetry), we find the ratios $BQ : BE$ and $CR : CF$:

$$\frac{BQ}{BE} = \frac{1-w}{1-w+vw},$$
$$\frac{CR}{CF} = \frac{1-u}{1-u+wu}.$$

Having found these ratios, we see next that

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABD} = \frac{1-v}{1-v+uv}.$$

We also know that $BD/BC = u$. From this it follows that:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABC} = u.$$

Hence by multiplication we get:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{u(1-v)}{1-v+uv}.$$

In the same way (or by using symmetry), we find that:

$$\frac{\text{Area of } \triangle BCQ}{\text{Area of } \triangle ABC} = \frac{v(1-w)}{1-w+vw},$$
$$\frac{\text{Area of } \triangle CAR}{\text{Area of } \triangle ABC} = \frac{w(1-u)}{1-u+wu}.$$

Hence we have:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = 1 - \frac{u(1-v)}{1-v+uv} - \frac{v(1-w)}{1-w+vw} - \frac{w(1-u)}{1-u+wu}.$$

This yields the desired formula

$$g(u,v,w) = 1 - \frac{u(1-v)}{1-v+uv} - \frac{v(1-w)}{1-w+vw} - \frac{w(1-u)}{1-u+wu}. \tag{13}$$

This result is generally known as Routh's theorem, named after Edward John Routh who first mentioned it in a book published in 1896. (However, it appears to have been known much before that date. It was used as an examination question in the famous Mathematical Tripos.)

We may verify after algebraic manipulation that $g(t,t,t) = f(t)$. If we define a new set of quantities x, y, z by:

$$x = \frac{BD}{DC}, \quad y = \frac{CE}{EA}, \quad z = \frac{AF}{FB},$$

then the statement of the result assumes a slightly more convenient form:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)}. \quad (14)$$

Pure geometry proof of Routh's theorem. Pure geometry proofs of the theorem have been known for some time. We conclude this article with one such proof which appeared in the magazine *Crux Mathematicorum*; see [2]. The proof is due to James S. Kline and Daniel J. Velleman, and is both ingenious and compact.

In Figure 5, we have drawn lines through P which are parallel to the sides of the triangle, thus giving rise to segments $GH \parallel BC$, $IJ \parallel CA$ and $KL \parallel AB$. (Note: we have suppressed the labels of points F, R, Q to avoid a visual clutter.) Our objective is to find the ratio of the area of $\triangle ABP$ to the area of $\triangle ABC$. Towards this end, we shall find the ratio JP/AC. We shall work in terms of x, y, z rather than u, v, w. (Recall that $BD/DC = x$, $CE/EA = y$ and $AF/FB = z$.) By triangle similarity, we have:

$$\frac{JP}{PI} = \frac{AE}{EC} = \frac{1}{y},$$

hence $PI = y \cdot JP$.

Again, $\triangle PJG \sim \triangle HKP$ and $\triangle AGH \sim \triangle ABC$, hence:

$$\frac{HK}{PJ} = \frac{HP}{PG} = \frac{CD}{DB} = \frac{1}{x},$$

hence $HK = JP/x$.

The expressions for the lengths of PI and HK now lead to the following:

$$\begin{aligned} CA &= CH + HK + KA \\ &= IP + HK + PJ \\ &= y \cdot JP + \frac{JP}{x} + JP. \end{aligned}$$

Hence we have:

$$\frac{JP}{AC} = \frac{1}{1 + 1/x + y} = \frac{x}{xy + x + 1}.$$

The ratio of the altitudes of $\triangle ABP$ and $\triangle ABC$ through P and C respectively must be given by the

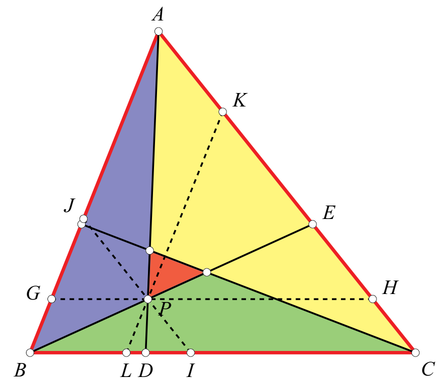


Figure 6.

same expression, $x/(xy + x + 1)$. As the two triangles share the same base AB, the ratio of the area of $\triangle ABP$ to that of $\triangle ABC$ is given by that very same expression:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{x}{xy + x + 1}.$$

We derived this expression by drawing lines through P which are parallel to the sides of $\triangle ABC$. By similarly drawing lines through Q and R respectively which are parallel to the sides of the triangle (we have not shown these lines), we derive expressions for the ratios of the areas of $\triangle BCQ$ and $\triangle CAR$ to that of $\triangle ABC$. We obtain the following:

$$\frac{\text{Area of } \triangle BCQ}{\text{Area of } \triangle ABC} = \frac{y}{yz + y + 1},$$

$$\frac{\text{Area of } \triangle CAR}{\text{Area of } \triangle ABC} = \frac{z}{zx + z + 1}.$$

Hence:

$$\begin{aligned} \frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} &= 1 - \frac{x}{xy + x + 1} \\ &\quad - \frac{y}{yz + y + 1} - \frac{z}{zx + z + 1} \\ &= \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)} \end{aligned}$$

The last step requires a bit of algebraic jugglery, which we leave you to undertake. For more on Routh's theorem, see [1] and [3].

References

1. M. S. Klamkin and A. Liu, "Three More Proofs of Routh's Theorem." *Crux Mathematicorum* **7**, 199-203, 1981
2. J. S. Kline and D. Velleman. "Yet another proof of Routh's theorem," *Crux Mathematicorum* **21**, 37-40, 1995
3. Routh's theorem, Wikipedia. https://en.wikipedia.org/wiki/Routh's_theorem

SHAILESH SHIRALI is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.