in the classroom

The Cost of Money

CoMaC

In a post that appeared early in 2014 on the site http://mathforlove.com/2014/02/a-dollar-that-costs-a-dollar/, a very interesting and mathematically elegant question was posed which we study here. Consider the US coinage system (coins only; we do not include any notes of value $1 or more). We have the following coins: Penny (1 cent, $ 0.01), Nickel (5 cents, $ 0.05), Dime (10 cents, $ 0.10) and Quarter (25 cents, $ 0.25). Each of these coins is made of some metal or mixture of metals, and as such each one has a cost associated with it. After all, the metals in question have to be purchased from the market! At the time the post was written, here were the relevant costs (the cost is given in cents per coin):

Coin	Penny	Nickel	Dime	Quarter
Market cost (cents)	2.5	11	6	11

The author asked the following questions of his students:
(a) Which mix of coins makes the cheapest dollar? (b) Which mix of coins makes the most expensive dollar? These questions are easy to answer and we do not study them. But a question was posed by one of the girls in the class which was mathematically far more rich:

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Can you make a dollar in coins that also ***costs*** *a dollar to make?*

Let's see what this questions involves. To make the dollar using only dimes, we need ten dimes; that would cost 60 cents. Not good enough. How about a dollar made up of twenty nickels? That costs $20 \times 11 = 220$ cents. No, that does not work either. How about eight dimes, two nickels, and ten pennies? That costs 95 cents. Close, but still not good enough! Can we find a combination whose cost is exactly \$1?

The author reports that the girl found a solution after an intense session lasting thirty-five minutes. He marvels not just at her creativity in asking such a rich question but also at her diligence in finding an answer: (He writes: "I had one of those awesome experiences this week where a student thinks of a better question." Would that we all have more such experiences!)

We shall address the same question here. But being math teachers we shall go a step further: we shall ask for *all* possible solutions.

Mathematical formulation. Let a, b, c, d denote the numbers of pennies, nickels, dimes and quarters used. Then the monetary value of this 'portfolio' is $a + 5b + 10c + 25d$ cents, and the market cost is $2.5a + 11b + 6c + 11d$ cents (using the cost data given above). Hence we must find non-negative integers a, b, c, d which satisfy the following system:

(1) $\quad a + 5b + 10c + 25d = 100,$

(2) $\quad 2.5a + 11b + 6c + 11d = 100.$

Our task is to find all the solutions to this system.

Equation (1) tells us that a is a multiple of 5 (because $5b$, $10c$, $25d$ and 100 are all multiples of 5), and equation (2) tells us that a is even. Hence a is a multiple of 10. Let $a = 10x$ where x is a non-negative integer. Substituting $a = 10x$ in both the equations and simplifying, we get:

(3) $\quad 2x + b + 2c + 5d = 20,$

(4) $\quad 25x + 11b + 6c + 11d = 100.$

From equation (3) we get $b = 20 - 2x - 2c - 5d$. Substituting for b in equation (4) we get

$25x + 11(20 - 2x - 2c - 5d) + 6c + 11d = 100,$
and therefore

(5) $\quad 3x - 16c - 44d + 120 = 0.$

From equation (5) we see that $3x$ is a multiple of 4 and hence that x is a multiple of 4. Let $x = 4y$ where y is a non-negative integer. (So the relation between a and y is: $a = 40y$.)

The system now becomes:

(6) $\quad 8y + b + 2c + 5d = 20,$

(7) $\quad 100y + 11b + 6c + 11d = 100.$

From equation (7) we see that y cannot exceed 1. Thus, y can only be 0 or 1.

If $y = 1$ then equation (7) implies that $b = 0$, $c = 0$, $d = 0$. But then equation (6) is not satisfied. So this possibility does not lead to a solution. Hence $y = 0$ (which means that $x = 0$ and hence also $a = 0$). This yields:

(8) $\quad b + 2c + 5d = 20,$

(9) $\quad 11b + 6c + 11d = 100.$

We now make use of the terms $11b$ and $11d$ in equation (9); they are (obviously) multiples of 11. Since $100 \equiv 1 \pmod{11}$, it follows that $6c \equiv 1 \pmod{11}$, and hence that c is one of the following numbers: 2, 13, 24, 35, ... Also from equation (8) we get $2c \le 20$, i.e., $c \le 10$. Hence $c = 2$. Therefore we get:

$$b + 5d = 16,$$

$$b + d = 8.$$

This pair of equations may be solved to yield $b = 6$, $d = 2$. So we obtain a solution:

$$(a, b, c, d) = (0, 6, 2, 2).$$

Please check that these values do satisfy the original equations. *This is the only solution possible*. That is, the only answer to the proposed problem is:

- 0 pennies;
- 6 nickels (worth \$0.30 and costing \$0.66);
- 2 dimes (worth \$0.20 and costing \$0.12);
- 2 quarters (worth \$0.50 and costing \$0.22).

References

1. http://mathforlove.com/2014/02/a-dollar-that-costs-a-dollar/
2. http://mathforlove.com/wp-content/uploads/2014/02/ How-Much-Does-Money-Cost-v2-simplification.pdf

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