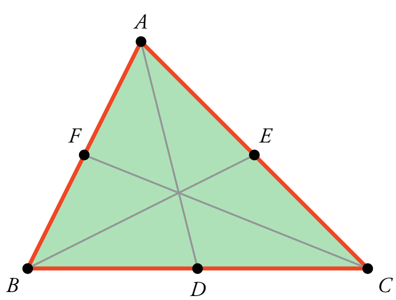
Addendum to

Integer-Sided Triangles with Perpendicular Medians

C⊗MαC

In the accompanying article, the author enquired into the conditions required on the sides a, b, c of $\triangle ABC$ in order to make its median triangle right-angled. We give an alternative treatment here.

Alternative proof of Lemma 1. The lemma states: *If $\triangle ABC$ is a triangle with medians AD, BE and CF, then there exists a triangle whose sides are respectively congruent to these three medians.*



Proof using vector algebra. Arbitrarily choose A to be the origin. Let the position vectors of B, C be $\overrightarrow{b}$, $\overrightarrow{c}$, respectively. Then the vectors representing medians AD, BE, CF are respectively:

$$\overrightarrow{AD} = \tfrac{1}{2}\overrightarrow{b} + \tfrac{1}{2}\overrightarrow{c}, \quad \overrightarrow{BE} = -\overrightarrow{b} + \tfrac{1}{2}\overrightarrow{c},$$
$$\overrightarrow{CE} = -\overrightarrow{c} + \tfrac{1}{2}\overrightarrow{b}. \tag{1}$$

Observe that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$ is the zero vector. Hence a triangle exists with sides that are congruent and parallel to AD, BE, CF respectively. □

Finding integer solutions to the equation $\mathbf{a}^2 + \mathbf{b}^2 = 5\mathbf{c}^2$: a different approach. Divide by c^2 and write $u = a/c$, $v = b/c$. Then u and v are rational numbers, and we have:

$$u^2 + v^2 = 5. \tag{2}$$

We must solve this equation over the rational numbers $\mathbb{Q}$.

Rewrite the equation as $u^2 - 4 = 1 - v^2$. Both sides yield to factorisation, and we get:

$$(u-2)(u+2) = (1-v)(1+v).$$

Hence:

$$\frac{u-2}{1+v} = \frac{1-v}{u+2} = t \ (\text{say}), \tag{3}$$

where t is some rational number. This yields:

$$u - 2 = t(1+v), \quad 1 - v = t(u+2),$$

i.e.,

$$u - tv = t + 2,$$
$$tu + v = 1 - 2t.$$

These two equations may be solved for u and v in terms of the parameter t; we get:

$$u = \frac{2\left(1 + t - t^2\right)}{1 + t^2}, \quad v = \frac{1 - 4t - t^2}{1 + t^2}. \tag{4}$$

Now write $t = n/m$ where n, m are coprime integers. We get:

$$u = \frac{2\left(m^2 + mn - n^2\right)}{m^2 + n^2},$$
$$v = \frac{m^2 - 4mn - n^2}{m^2 + n^2}. \tag{5}$$

Recalling the definitions of u, v as $u = a/c$ and $v = b/c$, we see that we may write:

$$a = 2k\left(m^2 + mn - n^2\right),$$
$$b = k\left(m^2 - 4mn - n^2\right),$$
$$c = k\left(m^2 + n^2\right),$$

for some rational number k. We have obtained a two-parameter solution which is identical to what had been obtained earlier.