ClassRoom

# An Astonishing Property of Third-Order Magic Squares

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In the previous issue of *At Right Angles*, it was shown that using the numbers from 1 to 9, there is essentially just one third-order magic square (here the phrase "essentially just one" actually means: "essentially just one, up to rotations and reflections"), namely:

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

In an article [1] written in the February 1999 issue of the *American Mathematical Monthly*, titled appropriately "Magic Squares Indeed!", the authors point out a truly remarkable property of this magic square; namely:

$$816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2,$$

$$834^2 + 159^2 + 672^2 = 438^2 + 951^2 + 276^2.$$

Please verify for yourself that these relations are true. In fact, there are even more such relations, but we will list them later. For the moment we ask: what makes these relations true? What we shall

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show is still more surprising: these sum-of-squares equalities are true not just for this particular square but for any third-order magic square!

In their article, Benjamin and Yasuda use linear algebra to prove their results. We shall do so using only elementary algebra, accessible even at the ninth standard level.

**Proof.** We showed in the earlier article that if $s$ is the magic sum of a third-order magic square, and $m$ is the number in the central cell, then $s = 3m$. This simple relation allows us to find a generating formula for third-order magic squares. Let $a$, $b$ be respectively the numbers in the two corner cells of the top row. Then we can express the numbers in all the other cells in terms of $a$, $b$, $m$, using the magic property of the square repeatedly (namely, that the sum of the numbers in each row, each column and each diagonal is $3m$). The result is shown below:

| | | |
|---|---|---|
| $a$ | $3m - a - b$ | $b$ |
| $m - a + b$ | $m$ | $m + a - b$ |
| $2m - b$ | $a + b - m$ | $2m - a$ |

It is easy to check that this is a magic square for any values of $a$, $b$, $m$. *Note that for the moment we drop the requirement that the entries of the square must be all different from one another and must all be positive integers.* We are treating the above array simply as an algebraic object with the property that its row sums, column sums and diagonal sums are all the same (equal to $3m$). For ease of notation, we denote the above magic square by $f(m, a, b)$. Here is an instance of a magic square generated by the above formula:

$$f(8,7,3) = \begin{array}{|c|c|c|}\hline 7 & 14 & 3 \\ \hline 4 & 8 & 12 \\ \hline 13 & 2 & 9 \\ \hline \end{array}$$

Here are two instances of magic squares generated by this formula, in which the requirement that the entries must be all different from one another and must all be positive integers does not hold:

$$f(2,3,4) = \begin{array}{|r|r|r|}\hline 3 & -1 & 4 \\ \hline 3 & 2 & 1 \\ \hline 0 & 5 & 1 \\ \hline \end{array} \qquad f(1,2,6) = \begin{array}{|r|r|r|}\hline 2 & -5 & 6 \\ \hline 5 & 1 & -3 \\ \hline -4 & 7 & 0 \\ \hline \end{array}$$

**Associating two polynomials with each third-order magic square.** Next, with the magic square $f(m, a, b)$ we associate two polynomials as follows.

- We first associate a quadratic polynomial with each row, using the entries in the cells as the coefficients. Thus, for the first row, we multiply the first number by $x^2$, the second number by $x$, and retain the third number as it is; then we add these three expressions together. We do exactly the same for the second row and for the third row. At the end, we have three quadratic expressions, one associated with each row. We square these and add the three resulting polynomials. The result (naturally) is a polynomial of degree 4.

  For example, for the magic square $f(8,7,3)$ shown above, the three rows yield the following three quadratics respectively:

  $$7x^2 + 14x + 3, \qquad 4x^2 + 8x + 12, \qquad 13x^2 + 2x + 9.$$

  On adding their squares, we get the following:

  $$\left(7x^2 + 14x + 3\right)^2 + \left(4x^2 + 8x + 12\right)^2 + \left(13x^2 + 2x + 9\right)^2,$$

which simplifies to the following fourth-degree expression:

$$234x^4 + 312x^3 + 636x^2 + 312x + 234.$$

- Now we do the same thing in reverse, starting from the third column and working our way towards the first column. Thus, for the first row, we multiply the third number by $x^2$, the second number by $x$, and retain the first number as it is; then we add these three expressions together. And we do exactly the same for the second row and for the third row. At the end, we have three more quadratic expressions, one for each row (but they are different from the ones obtained earlier). We square them and add the resulting polynomials. The result once again is a polynomial of degree 4.

  So, with the magic square $f(8, 7, 3)$, the three rows yield the following three quadratic polynomials respectively:

  $$3x^2 + 14x + 7, \quad 12x^2 + 8x + 4, \quad 9x^2 + 2x + 13,$$

  and on adding their squares, we get the following:

  $$\left(3x^2 + 14x + 7\right)^2 + \left(12x^2 + 8x + 4\right)^2 + \left(9x^2 + 2x + 3\right)^2,$$

  which when simplified yields the following fourth-degree expression:

  $$234x^4 + 312x^3 + 636x^2 + 312x + 234.$$

  What do we notice? Why, it is the very same polynomial as the one obtained earlier! How very striking, how very odd, how very pleasing. But would this be true in general? Have we uncovered a new property about magic squares?

The answer: **Yes**, and it is easy to prove. For the magic square $f(m, a, b)$, the three rows yield the following three pairs of quadratic polynomials respectively:

| | Polynomial I: Reading col 1 to col 3 | Polynomial II: Reading col 3 to col 1 |
|---|---|---|
| First row | $ax^2 + (3m - a - b)x + b$ | $bx^2 + (3m - a - b)x + a$ |
| Second row | $(m - a + b)x^2 + mx + (m + a - b)$ | $(m + a - b)x^2 + mx + (m - a + b)$ |
| Third row | $(2m - b)x^2 + (a + b - m)x + (2m - a)$ | $(2m - a)x^2 + (a + b - m)x + (2m - b)$ |

For each column ('Polynomial I', 'Polynomial II'), we add the squares of the terms in the three rows. The result is a fourth-degree polynomial with coefficients as follows (we have chosen to display the polynomial in this form as it has too many terms to display in a single line):

| | |
|---|---|
| Coefficient of $x^4$ | $2a^2 - 2ab - 2am + 2b^2 - 2bm + 5m^2$ |
| Coefficient of $x^3$ | $-2\left(a^2 + 2ab - 4am + b^2 - 4bm + m^2\right)$ |
| Coefficient of $x^2$ | $3\left(4ab - 4am - 4bm + 7m^2\right)$ |
| Coefficient of $x^1$ | $-2\left(a^2 + 2ab - 4am + b^2 - 4bm + m^2\right)$ |
| Coefficient of $x^0$ | $2a^2 - 2ab - 2am + 2b^2 - 2bm + 5m^2$ |

The crucial finding is: *the two polynomials are identical.* (Note the palindromic patterns in the coefficients. Note also that each of the five coefficients is symmetric in $a$ and $b$.)

With this result in our possession, it is easy to make sense of the finding reported by the two authors in the article mentioned at the start. For, the sums of the squares computed as described simply correspond to assigning the value $x = 10$ in the polynomials above.

**Closing remark.**

| | | |
|---|---|---|
| 8 | 1 | 6 |
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With reference to the 'standard' third-order magic square, the authors of [1] had drawn attention to still more such equalities, namely, the following:

$$\begin{aligned} \text{Diagonals:}\quad & 456^2 + 231^2 + 978^2 = 654^2 + 132^2 + 879^2, \\ \text{Counter-diagonals:}\quad & 639^2 + 174^2 + 852^2 = 936^2 + 471^2 + 258^2, \\ \text{Diagonals:}\quad & 654^2 + 798^2 + 213^2 = 456^2 + 897^2 + 312^2, \\ \text{Counter-diagonals:}\quad & 693^2 + 714^2 + 258^2 = 396^2 + 417^2 + 852^2. \end{aligned}$$

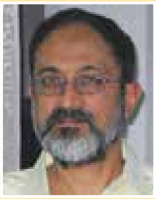
We invite the reader to show algebraically that any third-order magic square must possess the very same properties.

We also invite the reader to explore possible interpretations that can be given to our findings for particular values of $x$. For example, can any meaningful interpretation be given for the value $x = 1$? Or for the value $x = -1$?

Isn't it a truly wonderful exhibition of the power and reach of simple algebra that we have been able to demonstrate these properties of third-order magic squares?

**References**


1. A. Benjamin and K. Yasuda, "Magic Squares Indeed!", Amer. Math. Monthly, Feb. 1999. Available at https://www.math.hmc.edu/~benjamin/papers/kan.pdf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