ClassRoom

How To Prove It

In a previous issue of AtRiA (as part of the "Low Floor High Ceiling" series), questions had been posed and studied about polyominoes. In this article, we consider and prove two specific results concerning these objects, and make a few remarks about an open problem.

Shailesh A Shirali

Polyomino as a natural generalisation of a domino
We are familiar with the notion of a **domino**, which is a shape produced by joining two unit squares edge-to-edge. If we divide this object into two equal parts, we get a **monomino**. See Figure 1.

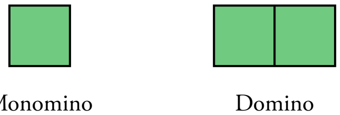


Figure 1. Monomino and domino

A polyomino is a natural generalisation of this notion, where we allow the number of unit squares to vary. For example, using three unit squares, the shapes shown in Figure 2 are possible. They are called **trominoes**.

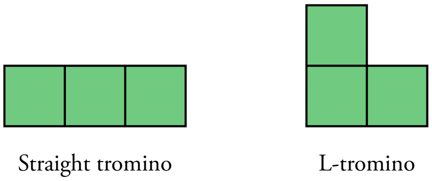


Figure 2. The two trominoes

Keywords: *Polyominoes, polygon, perimeter, parity, closed figure*

Similarly, by joining four unit squares edge-to-edge, the shapes shown in Figure 3 are possible. They are called **tetrominoes**. The names given to the individual shapes are shown alongside.

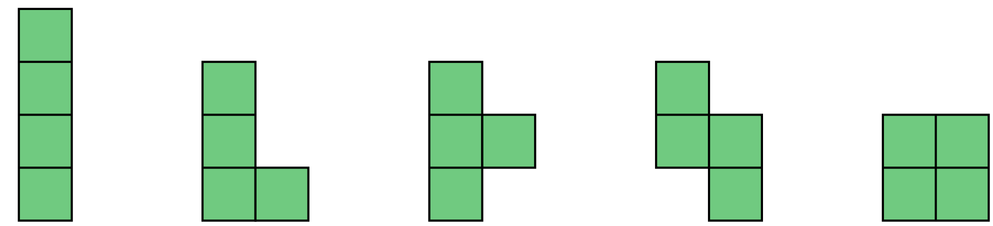


Figure 3. The five tetrominoes

For $n = 5$, the shapes are called **pentominoes**. It turns out that there are 12 possible pentominoes. We invite you to list them. For $n = 6$, the shapes are called **hexominoes**. As you may guess, there are a large number of these figures.

In general, we may define a polyomino as "a plane geometric figure formed by joining one or more equal squares edge-to-edge." (This is the definition given in [3]. See also [4]. For more about polyominoes, you may refer to the highly readable accounts in [1] and [2].)

As we increase the number of unit squares, more complex shapes become possible. For example, we may get shapes with 'holes' such as the one shown in Figure 4.

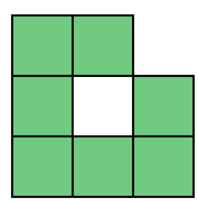


Figure 4. A 7-omino with a hole

Perimeter and number of sides of a polyomino

An n-omino is a plane geometric figure formed by joining n equal squares edge-to-edge. In this section we focus on two numbers associated with this object: its perimeter, and its number of sides. Note that by definition, the area of an n-omino is n square units. Table 1 lists the perimeter P and number of sides k for all polyominoes with area not exceeding 4.

Area, n	Shape	Perimeter, P	Number of sides, k
1	Monomino	4	4
2	Domino	6	4
3	Straight tromino	8	4
3	L-tromino	8	6
4	Straight tetromino	8	4
4	L-tetromino	10	6
4	T-tetromino	10	8
4	Z-tetromino	10	8
4	Square tetromino	8	4

Table 1

On examining the data, we see something noteworthy right away: *P and k are even numbers in every case.* Will this be the case for polyominoes with larger numbers of sides? We shall show that the answer is **Yes**. The proofs we offer are very instructive, as they use the notion of *parity*.

Proof that the perimeter P is even. The sides of the unit squares making up a polyomino give rise in a natural way to two mutually perpendicular directions which we may regard as a pair of coordinate axes. Let these axes be drawn; the outer boundary of a polyomino is now entirely composed of segments of unit length, each parallel to the x-axis or the y-axis.

Now let us take a walk around the outer boundary of the polyomino, advancing in steps of unit length and marking a dot at each lattice point along our path. (We must fix a direction for our tour; let us assume that we walk in the counterclockwise direction.) The number of steps we take clearly equals the perimeter P of the polyomino. With each step, our location changes in the following way: either the x-coordinate changes by 1 unit, or the y-coordinate changes by 1 unit; but not both at the same time. Hence if P_1 and P_2 are two successive lattice points along the path, then $P_1 - P_2$ is one of the following:

$$(1,0), \quad (-1,0), \quad (0,1), \quad (0,-1).$$

Let a, b, c, d be the respective number of steps of each of the above types, as we traverse the outer boundary. Then:

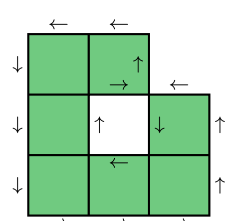
$$P = a + b + c + d.$$

After a full circuit, the total change in the x-coordinate must be 0; hence $a = b$. In the same way, we must have $c = d$, because the total change in the y-coordinate must be 0. Hence:

$$P = 2a + 2c = 2(a + c).$$

It follows that P is an even number. $\square$

Figure 5 illustrates the meanings of the parameters a, b, c, d for the polyomino depicted in Figure 4, the path being described by the arrows.



- $a = 4$, $c = 4$
- $b = 4$, $d = 4$

Figure 5. Traversing the outer boundary of the 7-omino shown earlier

Proof that the number of sides is even. Suppose that the polyomino is a k-sided polygon (or k-gon for short), with k vertices. (Note that the polygon need not be convex.) Let the vertices be $P_1, P_2, \ldots, P_k$. As we traverse the outer boundary in the counterclockwise direction, let the angle through which we need to turn at vertex P_i be θ_i (angles measured in degrees); then $\theta_i = \pm 90$ for each i. Let a be the number of vertices where we make a $+90^\circ$ turn, and let b be the number of vertices where we make a -90° turn; then $k = a + b$, and the total turning angle is $90(a - b)^\circ$.

Now, for any polygon, the total turning angle as we traverse the outer boundary is necessarily a multiple of 360°. In the case of a convex polygon, the total turning angle is exactly $\pm 360^\circ$, but for polygons with regions of non-convexity and/or holes, the total turning angle may be a higher multiple of 360°. Figure 6 shows examples: (a) where the total turning angle is 360°; (b) where the total turning angle is 720°.

(a) $(8 \times 90^\circ) - (4 \times 90^\circ) = 360^\circ$ (b) $8 \times 90^\circ = 720^\circ$

Figure 6. Total turning angle for a polyomino

From this reasoning, we deduce that $90(a-b)$ is a multiple of 360, and therefore that $a-b$ is a multiple of 4. Hence $a-b$ is an even number. Therefore, $a+b=(a-b)+2b$ is an even number as well. That is, k is even. So the number of sides of the polyomino is an even number. □

Unsolved problems

There is just 1 monomino, and just 1 domino; there are 2 trominoes, and 5 tetrominoes. These numbers give rise to an interesting but difficult problem. For any positive integer n, let $f(n)$ denote the number of different n-ominoes. Care is needed in interpreting the word 'different.' We regard two shapes as 'the same' if they are geometrically congruent to each other ('congruent' in the usual, Euclidean sense of that word); and two shapes are 'different' if they are not congruent to one another. With this understanding, we find the following values taken by the function f.

n	1	2	3	4	5	6	7	$\cdots$
$f(n)$	1	1	2	5	12	35	108	$\cdots$

The 12 possible pentominoes are depicted in Figure 7. We will leave it to you to sketch the 35 possible hexominoes. Or you may refer to [3] for some sketches.

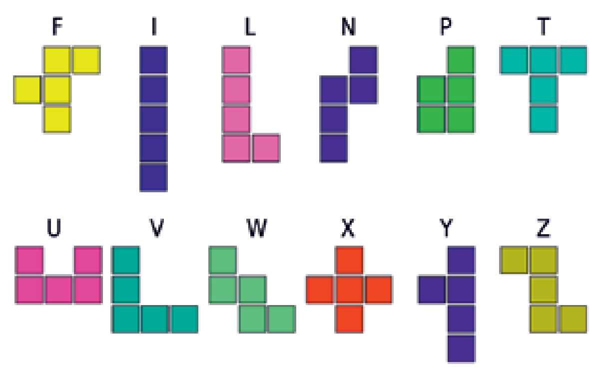


Figure 7. The 12 pentominoes
Source: https://simple.wikipedia.org/wiki/Pentomino#/media/File:Pentominos.svg.

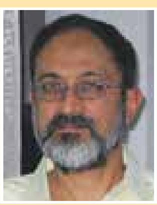
Now for the problems that these numbers suggest: (a) Given n, is there a simple way of computing the value of $f(n)$? (b) Is there a simple formula for $f(n)$? As of now, the answers to both questions appear to be **No**, and the only known way to compute $f(n)$ for higher values of n is to use computer-assisted enumeration, based on clever algorithms. Using such means, the sequence of values of f has been obtained to many terms (the first two values are $f(1)=1$ and $f(2)=1$):

1, 1, 2, 5, 12, 35, 108, 369, 1285, 4655, 17073, 63600, 238591, 901971, 3426576, 13079255, 50107909, 192622052, 742624232, 2870671950, 11123060678, 43191857688, 168047007728, 654999700403, 2557227044764, 9999088822075, 39153010938487, 153511100594603, ….

These two questions and others of their kind continue to remain unanswered at the present moment. However, it has been shown (see [1] and [4]) that $3.72^n < f(n) < 4.65^n$ for all positive integers n.

References

1. Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays*, 13th ed. New York: Dover, pp. 109-113, 1987
2. Gardner, M. "Polyominoes and Fault-Free Rectangles." Ch. 13 in Martin Gardner's New Mathematical Diversions from Scientific American. New York: Simon and Schuster, pp. 150-161, 1966.
3. Polyomino. https://en.wikipedia.org/wiki/Polyomino
4. Polyomino. http://mathworld.wolfram.com/Polyomino.html



SHAILESH SHIRALI is Director and Principal of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as an editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.