ClassRoom

20-30-130, A Trigonometric Solution

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The problem we study in this short note belongs to an extremely interesting class of geometrical problems. Typically, they deal with triangles with many lines drawn within them, intersecting at angles whose measures are an integer number of degrees. We are required to find the measure of some indicated angle. A common feature of these problems is that solutions using only pure geometry are difficult to find (though not impossible), and one is forced to fall back upon trigonometry. The present problem is of just this type.

Figure 1 shows $\triangle ABC$ with $\measuredangle A = 130^\circ$, $\measuredangle B = 30^\circ$ and $\measuredangle C = 20^\circ$. Point P is located within the triangle by drawing rays as shown from B and C, such that $\measuredangle PBC = 10^\circ$ and $\measuredangle PCB = 10^\circ$. Segment PA is then drawn. We are asked to find the measure of $\measuredangle PAC$.

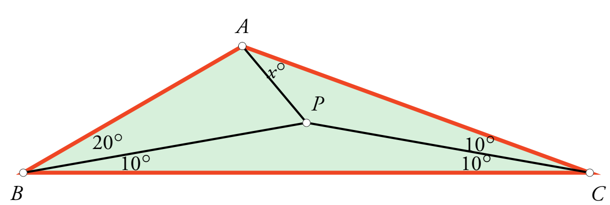

Figure 1

Keywords: *Integer degree, sine rule, complementary angle, supplementary angle, double angle identities, sine of the sum of two angles*

Solution. Let $\measuredangle PAC = x^\circ$; then $\measuredangle PAB = (130 - x)^\circ$. We now use the fact that $PB = PC$.

From $\triangle APB$ we have

$$\frac{AP}{PB} = \frac{\sin 20^\circ}{\sin(130 - x)^\circ}.$$

From $\triangle APC$ we have:

$$\frac{AP}{PC} = \frac{\sin 10^\circ}{\sin x^\circ}.$$

Hence:

$$\frac{\sin 20^\circ}{\sin(130 - x)^\circ} = \frac{\sin 10^\circ}{\sin x^\circ}.$$

This yields:

$$\begin{aligned} 2\cos 10^\circ &= \frac{\sin(130 - x)^\circ}{\sin x^\circ} \\ &= \sin 130^\circ \cot x^\circ - \cos 130^\circ \\ &= \sin 50^\circ \cot x^\circ + \cos 50^\circ. \end{aligned}$$

Hence:

$$\begin{aligned} \cot x^\circ &= \frac{2\cos 10^\circ - \cos 50^\circ}{\sin 50^\circ} \\ &= \frac{2\sin 80^\circ - \sin 40^\circ}{\cos 40^\circ} \\ &= 4\sin 40^\circ - \tan 40^\circ. \end{aligned}$$

Computation using a calculator shows that $4\sin 40^\circ - \tan 40^\circ \approx 1.732$. This *suggests* (but obviously does not prove) that $x = 30$.

Let us now formally prove that $x = 30$. For this, we must prove the following identity:

$$\tan 60^\circ = 4\sin 40^\circ - \tan 40^\circ,$$

i.e., $\tan 60^\circ + \tan 40^\circ = 4\sin 40^\circ$. Here, the LHS is equal to:

$$\begin{aligned} &\frac{\sin 60^\circ}{\cos 60^\circ} + \frac{\sin 40^\circ}{\cos 40^\circ} \\ &= \frac{\sin 60^\circ \cos 40^\circ + \sin 40^\circ \cos 60^\circ}{\cos 60^\circ \cos 40^\circ} \\ &= \frac{\sin 100^\circ}{\cos 60^\circ \cos 40^\circ} = \frac{2\sin 80^\circ}{\cos 40^\circ} \\ &= 4\sin 40^\circ, \end{aligned}$$

which is just what we wanted.

What we have actually shown is that $\cot x^\circ = \cot 30^\circ$. Since $0^\circ < x^\circ < 180^\circ$, it follows that $x = 30$. Hence $\measuredangle PAC = 30^\circ$. $\square$

Remark. Note the approach that we have used: trigonometric analysis, based on well-known properties of triangles; a guess at the answer, using approximate computations; and then a proof that the answer is correct, drawing on well-known identities. These are common themes in problems of this genre. We will see them again in the future.

The question of a pure geometry solution remains. We shall pass on the challenge to the reader. If you come up with a solution within the boundaries of synthetic geometry, do please share it with us!

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

List of trigonometric identities used in the article

To the reader: If you have 'insider knowledge' of trig identities, this article will validate many known facts. If you have not, then it may be a good idea for you to try and match which identity has been used in each instance in the article above.

Supplementary angle identities: For any angle x°,

$$\sin(180-x)^\circ = \sin x^\circ,$$

$$\cos(180-x)^\circ = -\cos x^\circ,$$

$$\tan(180-x)^\circ = -\tan x^\circ.$$

Double angle identities: For any angle x,

$$\sin 2x = 2\sin x\cos x,$$

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

Addition formula: For any two angles x and y,

$$\sin(x+y) = \sin x\cos y + \cos x\sin y,$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y,$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}.$$

Sine rule for triangles: For any $\triangle ABC$ with sides a, b, c:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the circumradius of the triangle.