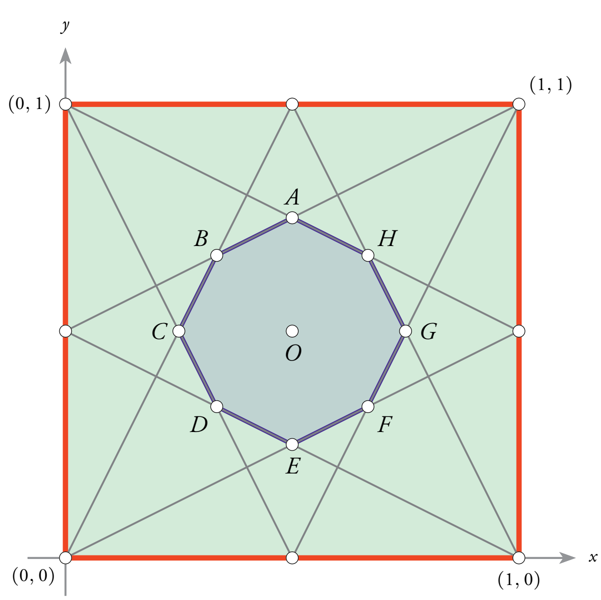
Octagon in a Square

ClassRoom

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

In the November 2015 issue of AtRiA, the following geometrical puzzle had been posed. An octagon is constructed within a square by joining each vertex of the square to the midpoints of the two sides remote from that vertex. Eight line segments are thus drawn within the square, creating an octagon (shown shaded). The following two questions had been posed: (i) Is the octagon regular? (ii) What is the ratio of the area of the octagon to that of the square?



Label the vertices of the octagon A, B, C, D, E, F, G, H as shown. Let O be the centre of the square. Recall that a polygon is said to be regular if its sides have equal length and its internal angles have equal measure. We shall show that the octagon is *not* regular!

We assign coordinates to the plane in such a way that the vertices of the enclosing square have coordinates $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, as shown. The slope of side AB is $1/2$, that of side BC is 2, and that of side CD is -2. Using a well-known formula from coordinate geometry (the one which gives the tangent of the angle between two lines, using their slopes), we compute the tangents of $\measuredangle ABC$ and $\measuredangle BCD$. We have:

$$\tan \measuredangle ABC = -\frac{2 - 1/2}{1 + 2 \cdot 1/2} = -\frac{3/2}{2} = -\frac{3}{4},$$

$$\tan \measuredangle BCD = -\frac{-2 - 2}{1 + (-2) \cdot 2} = -\frac{-4}{-3} = -\frac{4}{3}.$$

Note what we have found: the tangents of these two angles are not equal! It follows by symmetry that the tangents of the internal angles of the octagon are alternately $-3/4$ and $-4/3$. Hence the octagon is not regular. (For regularity, all the angles would obviously have to have the same tangent value.)

Something curious ...

We note something quite remarkable here. It is easy to see that each of the angles AOB, BOC, COD, ..., GOH, HOA equals 45°. Slightly less obvious but also true is the fact that the sides AB, BC, CD, ..., HA have equal length. For, the equations of the sides are the following:

$$\text{Equation of } HA: \quad y = -\frac{x}{2} + 1,$$

$$\text{Equation of } AB: \quad y = \frac{x}{2} + \frac{1}{2},$$

$$\text{Equation of } BC: \quad y = 2x,$$

$$\text{Equation of } CD: \quad y = -2x + 1.$$

Hence we get, by solving pairs of equations:

$$A = \left(\frac{1}{2}, \frac{3}{4}\right), \quad B = \left(\frac{1}{3}, \frac{2}{3}\right), \quad C = \left(\frac{1}{4}, \frac{1}{2}\right),$$

implying that:

$$AB^2 = \frac{1}{6^2} + \frac{1}{12^2} = \frac{5}{144},$$

$$BC^2 = \frac{1}{12^2} + \frac{1}{6^2} = \frac{5}{144}.$$

By symmetry it follows that all the eight sides of the octagon have equal length (equal to $\sqrt{5}/12$ of the sides of the square).

So we see here an instance of an octagon whose sides have equal length, and whose sides subtend equal angles at a point which is symmetrically placed relative to alternate sets of vertices of the octagon, yet which is not regular. Rather a counterintuitive result!

Remark. Another approach to showing that the octagon is not regular is: first compute the coordinates of the vertices (as done above), and then compute the distances from the centre O to the vertices. We have:

$$OA = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$OB = \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{2} - \frac{2}{3}\right)^2} = \frac{\sqrt{2}}{6}.$$

By symmetry we have $OA = OC = OE = OG = 1/4$ and $OB = OD = OF = OH = \sqrt{2}/6$. Since $1/4 \neq \sqrt{2}/6$, it follows that the octagon is not regular.

Computation of area

It remains to compute the area of the octagon relative to the area of the square. There are many ways of obtaining the result, but we shall simply use the sine formula for area of a triangle. We have, since $O = (1/2, 1/2)$:

$$OA = \frac{3}{4} - \frac{1}{2} = \frac{1}{4},$$

$$OB = \sqrt{\left(\frac{1}{3} - \frac{1}{2}\right)^2 + \left(\frac{2}{3} - \frac{1}{2}\right)^2} = \sqrt{\frac{1}{6^2} + \frac{1}{6^2}} = \frac{\sqrt{2}}{6}.$$

Hence:

$$\text{Area of } \triangle OAB = \frac{1}{2} \times \frac{1}{4} \times \frac{\sqrt{2}}{6} \times \sin 45^\circ = \frac{1}{2} \times \frac{1}{4} \times \frac{\sqrt{2}}{6} \times \frac{1}{\sqrt{2}} = \frac{1}{48}.$$

It follows that the area of the octagon is

$$8 \times \frac{1}{48} = \frac{1}{6}.$$

Hence the area of the octagon is $1/6$ of the area of the square.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.