Problem Corner

Problems for the Senior School

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PROBLEMS FOR SOLUTION

Problem V-1-S.1
What is the greatest possible perimeter of a right-angled triangle with integer sides, if one of the sides has length 12?

Problem V-1-S.2
Rectangle $ABCD$ has sides $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\measuredangle BPC = 90^\circ$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

Problem V-1-S.3
Let a, b, c be such that $a + b + c = 0$, and let

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}.$$

Determine the value of P.

Problem V-1-S.4
In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\measuredangle BAE = 40^\circ$. If $\measuredangle DAE = \measuredangle DFE$, what is the magnitude of $\measuredangle ADF$ in degrees?

Problem V-1-S.5
Circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume that $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\measuredangle PYX$ in degrees?

SOLUTIONS OF PROBLEMS IN ISSUE-IV-3 (NOVEMBER 2015)

Solution to problem IV-3-S.1
Determine all possible integers N such that $N(N-101)$ is the square of a positive integer.

Let $d = \text{GCD}(N, N-101)$. Then d divides $N-(N-101) = 101$. Hence $d = 1$ or $d = 101$.

If $d = 1$, then $N(N-101)$ is a perfect square if and only if each of N and $N-101$ is a square. Let $N = a^2$ and $N - 101 = b^2$. Then

$$(a-b)(a+b) = 101,$$

so $(a-b, a+b) = (1, 101)$, leading to $a = 51$, $b = 50$ and $N = 51^2 = 2601$.

If $d = 101$, then $N = 101k$ and $N - 101 = 101(k-1)$ for some positive integer $k > 1$. Therefore:

$$N(N-101) = (101^2)k(k-1).$$

But $k(k-1)$ is never a square for any positive integer $k > 1$, because

$$(k-1)^2 < k(k-1) < k^2.$$

Thus $N(N-101)$ is not a square when $d = 101$. It follows that there is just one integer value of N for which $N(N-101)$ is a perfect square; namely: $N = 2601$.

Solution to problem IV-3-S.2
Let R, S be two cubes with sides of lengths r, s, respectively, where r and s are positive integers. Show that the difference of their volumes numerically equals the difference of their surface areas if and only if $r = s$.

For the given condition to hold we must have

$$r^3 - s^3 = 6(r^2 - s^2). \qquad (1)$$

This can be expressed as

$$s^2 = (6-r)(r+s), \qquad (2)$$

which shows that $r < 6$. Similarly $s < 6$. Now observe that equation (1) can be written as

$$(6-r)^2 + (6-s)^2 + (r+s)^2 = 72. \qquad (3)$$

The right-hand side of relation (3) is divisible by 4. Therefore the left-hand side must be divisible by 4. The square of any positive integer is either divisible by 4, or exceeds a multiple of 4 by 1. Thus, in this case, each term on the left-hand side must be divisible by 4, which forces each term to be even. Hence, both r and s are even. If r, s are different, then r, s are 2, 4 in some order, so $r + s = 6$, and by equation (2),

$$s^2 = 6(6-r),$$

which is false for positive integers $r, s \in \{2, 4\}$. Thus $r = s$. If we know that $r = s$, then equation (1) is clearly true.

Solution to problem IV-3-S.3
Suppose $S = \{0, 1\}$ has the following addition and multiplication rules: $0+0 = 0$, $0+1 = 1+0 = 1$, $1+1 = 0$, $0 \times 0 = 1 \times 0 = 0 \times 1 = 0$, $1 \times 1 = 1$. A system of polynomials is defined with coefficients in S. Show that in this system $x^3 + x + 1$ is not factorisable.

Suppose the polynomial can be factored as $(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \{0, 1\}$. By equating the coefficients of $x^3, x^2, x^1, x^0 = 1$ on the two sides, we see that:

$$ac = 1, \quad bc + ad = 0, \quad bd + ae = 1, \quad be = 1.$$

Thus $a = b = c = e = 1$, which leads on substitution to $1 + d = 0$ and $d + 1 = 1$, which contradict each other. Hence the factorisation is not possible.

Solution to problem IV-3-S.4
Consider all non-empty subsets of the set $\{1, 2, 3, \ldots, n\}$. For each such subset, find the product of the reciprocals of each of its elements. Denote the sum of all these products by a_n. Prove that $a_n = n$ for all positive integers n.

Observe that $a_1 = 1$ and $a_2 = 2$. Suppose $a_k = k$ for some positive integer $k > 1$. Then:

$$a_{k+1} = \left(1 + \frac{1}{k+1}\right) a_k + \frac{1}{k+1}$$
$$= \frac{(k+1)^2}{k+1} = k + 1.$$

Thus by the principle of mathematical induction, $a_n = n$ for all natural numbers n.

Solution to problem IV-3-S.5

Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real zero.

Let $f(x) = x^8 - x^7 + x^2 - x + 15$. Observe that all the coefficients of $f(-x)$ are positive. Thus $f(x)$ does not have any real negative zero. We can write

$$f(x) = x^7(x - 1) + x(x - 1) + 15,$$

hence $f(x) > 0$ for $x \geq 1$. Writing $f(x)$ as

$$f(x) = x^8 + (1 - x^7) + x^2 + (1 - x) + 13,$$

we see that $f(x) > 0$ when $0 \leq x \leq 1$. Thus $f(x)$ does not have any real positive zero.