Problems for the Middle School

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Problem Corner

PROBLEMS FOR SOLUTION

Problem V-1-M.1
Find two non-zero numbers such that their sum, their product and the difference of their squares are all equal.

Problem V-1-M.2
Prove that a six-digit number formed by placing two consecutive three-digit positive integers one after the other is not divisible by any of the following numbers: 7, 11, 13.

(Adapted from the Mid-Michigan Olympiad in 2014 grades 7–9)

Problem V-1-M.3
If n is a whole number, show that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5.

Problem V-1-M.4
We know that the sum of two consecutive squares can be a square. For example, $3^2 + 4^2 = 5^2$.

a. Show that the sum of any m consecutive squares cannot be a square for $m \in \{3, 4, 5, 6\}$.
b. Can the sum of 11 consecutive square numbers be a square number?

Problem V-1-M.5

a. Which positive integers have exactly two positive divisors? Which have three positive divisors?

b. Among integers a, b, c, each exceeding 20, one has an odd number of divisors, and each of the other two has three divisors. If $a + b = c$, find the least value of c.

Problem V-1-M.6

A group of 43 devotees consisting of ladies, men and children went to a temple. After a ritual, the priest distributed 229 flowers to the visitors. Each lady got 10 flowers, each man got 5 flowers and each child got 2 flowers. If the number of men exceeded 10 but not 15, find the number of women, men and children in the group.

Problem V-1-M.7

There are two towns, A and B. Person P travels from A to B, covering half the distance at rate a, and the remaining half at rate b. Person Q travels from A to B (starting at the same time as P), travelling for half the time at rate a, and for half the time at rate b. Who reaches B earlier?

SOLUTIONS OF PROBLEMS IN ISSUE-IV-3 (NOVEMBER 2015)

Solution to problem IV-3-M.1

A number when increased by its cube results in the number 592788. *Find the number.*

Let the number be x; then $x + x^3 = 592788$. Since $x > 0$, we have

$$x^3 < x + x^3 < (x+1)^3,$$

i.e., $x^3 < 592788 < (x+1)^3$. So it suffices to find a pair of consecutive cubes between which 592788 lies. We find that it lies between 84^3 and 85^3, for we have $84^3 = 592704$ and $85^3 = 614125$. Hence $x = 84$, i.e., the required number is 84.

Solution to problem IV-3-M.2

Find the two prime factors of 206981 *given that one of them is approximately three times the other.*

Let the prime factorisation of 206981 be $206981 = pq$, where $q > p$; then q is close to $3p$. Hence we have: $p \times 3p \approx 206981$, i.e., $3p^2 \approx 206981$, which yields $p^2 \approx 68993$. Hence p is close to the square root of 68993, i.e., p is close to 263. It so happens that 263 is a prime number, and moreover that the quotient $206981/263$ is an integer; indeed, $206981/263 = 787$, and 787 is a prime number! So the required primes are 263 and 787.

Solution to problem IV-3-M.3

How would you distribute 44 *pencils to* 10 *students such that each student receives a different number of pencils?*

If we assume that at least one pencil is given to each student, then we must have at least $1 + 2 + 3 + \cdots + 10 = 55$ pencils. But we have only 44 pencils. Hence such a distribution is impossible. So there must be a student who does not get any pencil. But this requires at least $0 + 1 + 2 + \cdots + 9 = 45$ pencils. Hence even this distribution is impossible.

Solution to problem IV-3-M.4

Find the sum of all three-digit numbers $\overline{ABC}$ such that the two-digit numbers $\overline{AB}$ and $\overline{BC}$ are both perfect squares. [Jamaican Math Olympiad 2015]

The numbers $\overline{AB}$, $\overline{BC}$ must belong to the set $\{16, 25, 36, 49, 64, 81\}$. Hence $(\overline{AB}, \overline{BC})$ is one of the following: $(16, 64)$; $(36, 64)$; $(64, 49)$; $(81, 16)$. Hence the possible values of $\overline{ABC}$ are: 164, 364, 649, 816. So the required sum is $164 + 364 + 649 + 816 = 1993$.

Solution to problem IV-3-M.5

The numbers $1, 2, 3, 5, 7, 11, 13$ *are written on a board. You may erase any two numbers a, b and replace them by the single number $ab + a + b$. After repeating this process several times, only one number remains on the board. What might be this number? [Adapted from UAB MTS: 2006-2007]*

Note that $a + b + ab = (a+1)(b+1) - 1$; e.g., $2 + 5 + (2 \times 5) = (2+1)(5+1) - 1$. Thus, regardless of the order in which the numbers are selected, the result is the product of all the listed numbers increased by 1, from which then 1 is then subtracted. Hence the answer is:

$$(1+1)(2+1)(3+1)(5+1)(7+1)(11+1)$$
$$(13+1)-1 = 2\times 3\times 4\times 6\times 8\times 12$$
$$\times 14 - 1 = 193535.$$

Solution to problem IV-3-M.6

Between 3 PM and 4 PM, Ramya looked at her watch and noticed that the minute hand was between 5 and 6. When she looked next at the watch, slightly less than two hours later, she noticed that the hour and minute hands had switched places. What time was it when she looked at the watch the second time? [Adapted from "Mathematical Wrinkles" by S.I. Jones, 1912]

Suppose the times when she looked at the watch are x minutes past 3 PM and y minutes past 5 PM, respectively. Let us measure angles in degrees from the 12 o'clock position, in the clockwise direction. The interval between the events is given to be less than two hours long, so $y < x$.

On the first occasion when she looks at the watch, the angles made by the hour hand and the minute hand relative to the 12 position are, respectively:

$$90 + \frac{x}{2}, \quad 6x.$$

On the second occasion when she looks at the watch, the angles made by the hour hand and the minute hand relative to the 12 position are, respectively:

$$150 + \frac{y}{2}, \quad 6y.$$

Since the hour hand and the minute hand have exchanged positions, we have:

$$90 + \frac{x}{2} = 6y,$$
$$150 + \frac{y}{2} = 6x.$$

We solve this pair of equations simultaneously for x and y. Doing so in the standard manner (details omitted), we get:

$$x = \frac{3780}{143} = 26\tfrac{62}{143}, \quad y = \frac{2460}{143} = 17\tfrac{29}{143}.$$

Hence the time when she looks at the clock the second time is $5:17\frac{29}{143}$ PM, i.e., $17\frac{29}{143}$ minutes past 5 PM.