Two Problems

C⊗MαC

We present as earlier a small collection of problems, followed by their solutions. We state the problems first so you have a chance to try them out on your own.

PROBLEMS

1. Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ touches the smaller circle ω at Q. Show that PQ bisects $\measuredangle APB$. What might be a meaningful generalisation of this result? (See Figure 1.)

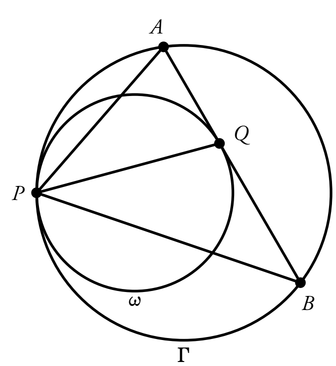

Figure 1. Two tangent circles

2. Show that the function $f(x, y) = \frac{1}{2}\left((x+y)^2 + x + 3y\right)$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ to $\mathbb{N}_0$ bijectively. (Here $\mathbb{N}_0$ refers to the set of all possible non-negative integers. So the problem asks us to show that as x and y take all possible non-negative integral values, $f(x, y)$ takes all possible non-negative integral values, with no repetition.)

Problem 1. *Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ touches the smaller circle ω at Q. Show that PQ bisects $\measuredangle APB$. What might be a meaningful generalisation of this result?*

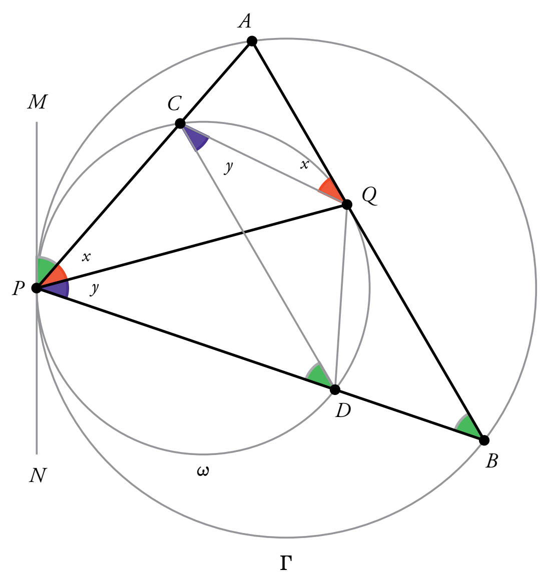

Figure 2

Solution. Let C be the point where PA intersects ω again, and let D be the point where PB intersects ω again (see Figure 2). Join CQ, DQ, CD as shown. The solution will now follow from old-fashioned 'angle chasing.' The two angles marked y are equal ("angles in the same segment of a circle"). So are the two angles marked x (this follows from the well-known theorem about the angle between a tangent to a circle and a chord at the point of contact).

Next we show that CD is parallel to AB. Here is one way of seeing why. Draw the tangent MN at P to ω; it is also the tangent at P to Γ. Using the theorem just quoted above, we see that $\measuredangle APM$ is equal to $\measuredangle CDP$ as well as to $\measuredangle ABD$. It follows that $\measuredangle CDP = \measuredangle ABP$, and hence that $CD \parallel AB$. From the parallelism, it follows that $\measuredangle AQC = \measuredangle DCQ$, i.e., $x = y$. Hence PQ bisects $\measuredangle APB$.

Another way of seeing why $CD \parallel AB$ uses the idea of a geometrical transformation, in this case a dilation centred at P, with scale factor $PA : PC$. The dilation maps ω to Γ and hence C to A and D to B; this implies that $AB \parallel CD$. $\square$

We have been asked to suggest a meaningful generalisation of the above result. Here is one which seems particularly attractive:

> *Two circles Γ and ω touch internally at P. A chord AB of the larger circle Γ cuts ω at points Q and R. Show that $\measuredangle APQ = \measuredangle BPR$.* (See Figure 3.)

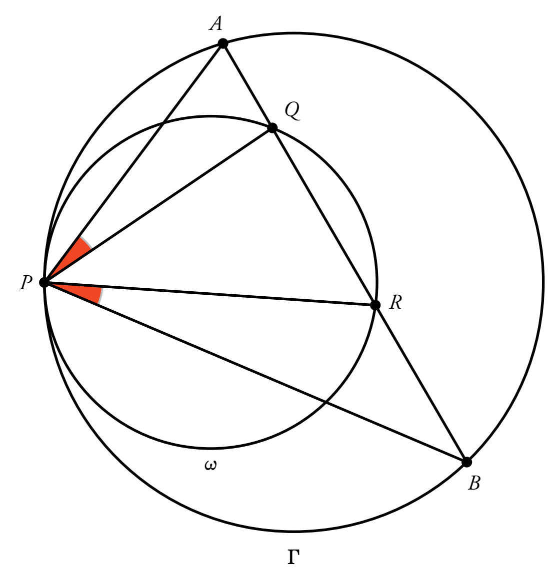

Figure 3. Generalisation of the earlier result

Do you see how this is a generalisation of the earlier result? See if you can find the proof for yourself!

Problem 2. *Show that the function $f(x, y) = \frac{1}{2}\left((x+y)^2 + x + 3y\right)$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ to $\mathbb{N}_0$ bijectively.* ($\mathbb{N}_0$ refers to the set of all non-negative integers.)

Solution. There are actually two parts to this question. Firstly, we must show that the range of f is the whole of the co-domain; i.e., for every non-negative integer n, there exists a pair x, y of non-negative integers such that $f(x, y) = n$. Secondly, we must show that two different pairs of non-negative integers cannot be mapped by f to the same image point. Only if we show both these parts can we claim that f is a bijective map. What we shall do now is to establish these two parts separately.

The expression for f can be written in the following form:

$$\begin{aligned} f(x, y) &= \frac{1}{2}\left((x+y)^2 + x + y\right) + y \\ &= \frac{(x+y)(x+y+1)}{2} + y. \end{aligned}$$

Let $x + y = k$; then $f(x, y) = T_k + y$, where T_k is the k-th triangular number. Since $y \leq x + y$, we must have $y \leq k$. So what we have to prove is the following: if n is any non-negative integer, then we can find a unique pair of integers k, y such that $0 \leq y \leq k$ and $n = T_k + y$.

We first show that such a pair of non-negative integers can always be found; i.e., every non-negative integer lies in the range of f. Our proof is algorithmic: given n, we show how to find x, y such that $f(x, y) = n$. All we do is to find the largest non-negative integer k such that $T_k \leq n$; then we let $y = n - T_k$ and $x = k - y$, and with this choice we have $f(x, y) = n$. An example will illustrate the mechanism. Let $n = 50$; since $T_9 < 50 < T_{10}$ and $50 - T_9 = 5$, we get $y = 5$ and $x = 9 - 5 = 4$. Check:

$$f(4, 5) = \frac{(4+5)^2 + 4 + 15}{2} = \frac{81 + 19}{2} = 50.$$

To complete this part of the proof, we must show that $y \leq k$. But this is clear, since $T_{k+1} - T_k = k + 1$, which implies that if $T_k \leq n < T_{k+1}$, then $n - T_k \leq k$.

Now for the uniqueness part, we must show that two different pairs of non-negative integers cannot map to the same value. For this it suffices to show the following: if a, b and c, d are pairs of integers such that

$$0 \leq b \leq a, \qquad 0 \leq d \leq c, \qquad T_a + b = T_c + d,$$

then $(a, b) = (c, d)$. To see why this is so, we treat a, c as fixed, and b, d as variables, with $0 \leq b \leq a$ and $0 \leq d \leq c$. If $a = c$ and $b \neq d$, then clearly $T_a + b \neq T_c + d$. So we may assume that $a \neq c$. Without loss of generality, suppose that $a < c$. The range of values taken by $T_a + b$ for $0 \leq b \leq a$ is the set

$$\{T_a, T_a + 1, T_a + 2, \ldots, T_a + a\},$$

and the range of values taken by $T_c + d$ for $0 \leq d \leq c$ is the set

$$\{T_c, T_c + 1, T_c + 2, \ldots, T_c + c\},$$

and these sets are clearly disjoint, because

$$T_a + a < T_{a+1} \leq T_c.$$

The stated claim follows. $\square$