A Review of

Taming the Infinite

By Ian Stewart

Review

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At some point students begin to show interest in the origins of mathematical ideas: How were logarithms discovered? Who was the first person to compute the log tables? When were quadratic equations first used, and how were they solved? Can equations of a higher order be solved in a similar way? Why are representations of complex numbers on a plane called Argand diagrams? Ian Stewart's book *Taming the Infinite: The Story of Mathematics from the First Numbers to Chaos Theory* throws light on questions like these. It is an attempt at mapping the major themes in mathematics through a historical perspective. This compact book of less than 400 pages covers the major topics in Mathematics and is accessible to students in secondary school. The latter part of the book gives a flavour of some areas of college mathematics. An underlying theme of the book is that the modern world owes a great debt to advances in Mathematics, and mention is made of some of the applications of Mathematics. Brief biographical sketches of the major players are given, though there are some odd omissions like Euler and Leibniz. There are boxed items, some with intriguing titles: *What We Don't Know About Primes, What Trigonometry Did For Us*. Though the book is compact, it is ambitious in its scope; but appears thin

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in some places as a result. A major drawback is that the book does not give much space to non-European strands: Aryabhata, Brahmagupta, Mahavira and Bhaskaracharya are given less than a couple of pages, and Madhava and the Chinese mathematicians do not even figure in the book.

Ian Stewart has tried to order the book thematically and chronologically, but this brings in a few anomalies. Chapter I on Numbers and Chapter III on Number Systems should sit together, but the chronology results in a chapter on Geometry in-between. The book starts by looking at the way numbers developed in two early civilizations – Egyptian and Babylonian. The Babylonians were able to accurately predict celestial events such as solar eclipses and the movements of planets. However, there is evidence that human beings were interested in counting much before these civilizations emerged. Two bones found in Africa, the Lebombo bone and Ishango bone, with tally marks on them, have got mathematicians speculating on their possible origins. Chapter III picks up the story with the background to our present number system which originated in India. A mention is made of some Indian Mathematicians during this period, and the legend of how Bhaskaracharya named one of his books (*Lilavati*) is nicely narrated. This section would have benefited from greater detail; those interested should refer to *The Crest of the Peacock* (reviewed in an earlier issue). Stewart gives a sketch of the life of Leonardo of Pisa, better known as Fibonacci, who introduced the Indian numeration system to Europe through his book *Liber Abbaci*.

Chapter II on Geometry begins with the Babylonians, who were familiar with what is now known as the Pythagorean Theorem. Mention is made of Archimedes in the context of the problem of finding good rational approximations to pi, and of finding formulas for the volume and surface area of a sphere. There is a biography of Hypatia, the woman mathematician who met a tragic end at the hands of a Christian mob in Alexandria. Euclid's *Elements*, and the concepts of *theorem, proof, postulate* and *axiom* are explained lucidly with examples; it will help the readers in putting the school geometry in context.

Chapter IV looks at the development of Algebra. Some readers will find it surprising that quadratic equations were being solved by the Babylonians in much the same way we do today. Stewart speculates that the Babylonians may have come upon this solution by depicting the equation pictorially – a form which students will find appealing. Next, Stewart takes us through a tour of attempts to solve the cubic equation, starting with attempts by the Greeks using the conics sections; much later, the same approach was formalised by the Persian Omar Khayyam, author of the poem *Rubiyat*. In the middle of the 16th century, an algebraic solution was found by Italian mathematicians. Readers may be surprised to find that many symbols in common use today were not in use at that time. For example, the '+' and '–' symbols appeared only around the 16th century. The 'equals to' sign was invented in 1557 by the English mathematician Robert Recorde, who said that he could not think of any two things more alike than a pair of parallel lines!

Ian Stewart gives the following example from an algebra book written in the 16th century (*Ars Magna*):

qdrat actur 4 rebus p: 32.

In modern notation this would be written as:

$$x^2 = 4x + 32.$$

Chapter V looks at how trigonometry and logarithms developed. Trigonometrical tables were developed in different ways in many places over the years; they arose from the needs of astronomers. The first trigonometric tables were derived by Hipparchus around 150 BC. In India, Aryabhata and Brahmagupta developed trigonometric concepts using the notion of the half-chord in a circle. The Arab mathematician Nasir-Uddin combined trigonometry in the sphere and the plane. This is one area where the mathematics done by the ancients is more complex than that learnt in school today. Plane trigonometry as currently taught in schools developed around the 15th century. On the other hand, logarithms developed from the need to make calculations easy; however, this aspect of logarithms has now become obsolete. The enormous effort put in by John Napier in constructing the logarithm tables eased the work of mathematicians and scientists. The number *e* which is closely related to logarithms makes an appearance here, though no one is credited with its discovery.

Today we take the idea of latitude and longitude for granted, but it took a long time for such ideas and for the notion of coordinate geometry to develop. In the next chapter, Stewart begins with Fermat who in the 17th century was the first to try and relate a geometric curve to an equation and to represent it on an oblique coordinate grid. The modern rectangular coordinate system was introduced by Descartes in an appendix to his book *Discourse de la Methode*; he showed how geometric curves can be represented using algebraic equations. Fermat extended the coordinate system to three dimensions. Jacob Bernoulli developed the idea of polar coordinates, where position is defined using an angle and a radial distance. The chapter ends by looking at some applications of coordinate geometry, including the use of GPS which is ubiquitous in today's world.

Chapter VII looks at Number Theory, a topic dealt with superficially in school mathematics. The early mathematicians mentioned here are Euclid and Diophantus; the absence of Asian mathematicians is glaring. Euclid proved many properties of primes; for example, that every number can be expressed as a product of prime numbers in a unique way. Stewart illustrates this with an example in a note, *Why Uniqueness Of Primes Is Not Obvious*.

The next big idea was the invention of Calculus, and it had an enormous impact on the development of mathematics. It seems to have arisen out of many unrelated investigations: instantaneous change in velocity, finding maxima and minima, finding tangents to curves, finding areas of planar shapes. The breakthrough was made by Leibniz, who was the first to realise that finding tangents to a curve was the inverse operation to finding area under the curve. He was the one who gave us nearly all the symbols we now associate with calculus. He published his work on calculus in 1684, though not many understood the significance of his work at the time. Newton had been developing his ideas on calculus at nearly the same time, but using different symbols; he published his ideas in 1687. Stewart gives a vivid description of the controversy that erupted regarding who was the "first to discover calculus", and the sizeable rift it caused between European and British mathematicians. The development of the planetary laws of motions, which can be regarded as a prime mover behind the development of calculus, is explained in detail, from Ptolemy's system of epicycles based around a stationary earth, to the Copernican sun-centred solar system, to the observations of Brahe and Kepler which allowed Kepler to come up with his laws of planetary motion; all these together with Galileo's work laid the foundations for calculus.

Calculus in its early days lacked a clear logical foundation, as the idea of limits had not been developed; that concept would take another century to develop. Leibniz used the term *infinitesimal* to describe a number close to zero, and Newton came up with the term *fluxion*. Since calculus bestowed such power in the hands of its users, most mathematicians ignored this lack of rigour. A significant exception was Bishop Berkeley who pointed out that it was meaningless to divide by a quantity that is later set equal to zero. How this conundrum was later resolved is described in chapter XI.

The next chapter looks at a topic familiar to students: Complex Numbers. Stewart explains that the early mathematicians ignored answers involving the root of a negative number. The first recorded manipulation of imaginary numbers is by Rafael Bombelli in his book *L'algebra* (1557). In the context of finding cube roots, he came across imaginary numbers and operated on them as if they were ordinary numbers. In 1673, John Wallis invented a way to represent imaginary numbers on a plane. Wallis's work was explained more clearly by Caspar Wessel in 1797, but his work went by unnoticed. The French mathematician Jean Robert Argand came up with the same idea independently in 1806; today we call his way of representing complex numbers *Argand diagrams*.

This was the start of the development of the field of complex analysis; ultimately it led to Cauchy's theorem which extends calculus to the complex plane; Gauss had come upon the same idea earlier but had not published it.

The rest of the book – apart from the chapter on Probability and Statistics – is devoted to topics that students study in college. Stewart has made a great effort at making the key ideas accessible to high school students. There are three chapters on non-Euclidean geometry. One such geometry which emerged from the work of the Renaissance artists is *projective geometry*. The search for a proof of Euclid's fifth postulate ultimately resulted in the development of a different kind of geometry. Gauss was convinced from an early stage that it ought to be possible to come up with a logically consistent system of non-Euclidean geometry (the geometry of curved space), but he was fearful of publishing his work as he felt that people were too conditioned to the geometry of Euclid and would ridicule his work. The chapter *Rubber Sheet Geometry* considers operations not studied in traditional geometry. For example, a shape like a square can be bent, stretched and twisted into a triangle or circle; likewise a coffee cup can be moulded into a doughnut shape. These objects which can be moulded into each other can be regarded as congruent, the only invariant aspect being *connectedness*. Stewart mentions the Königsberg bridges problem and discusses notions like the Möbius band and the Riemann Sphere.

The chapter titled *The Shape of Logic* deals with matters which are central to mathematics; it looks at how mathematicians started questioning the very foundation of mathematics. Dedekind was unsettled by the fact that obvious properties of real numbers had not been proved, for example, $\sqrt{2}\times\sqrt{3}=\sqrt{6}$, and he expressed his thoughts in a book he published in 1872. In another book in 1888, he exposed gaps in the foundations of the system of real numbers and proposed a new approach: Dedekind cuts. He found a way of defining the properties of real numbers solely in terms of rational numbers. This led to the question: How do we know that the properties for numbers hold true? In 1889, Giuseppe Peano proved the basic operations of arithmetic by creating a list of axioms for whole numbers, the most important ones being that there exists a whole number, 0, and each number n has a successor $s(n)$, which is 1 more than the previous one. As mathematicians grappled with these ideas, they began to explore the meaning of 'number'. In this context, sets were introduced as a 1-1 correspondence with numbers by Gottlob Frege. Unfortunately for Frege, his work was rendered worthless by George Cantor's work, and Bertrand Russell pointed out a paradox in his work just as his book was being published. Russell tried to fill the gap in Frege's work with his *theory of types*, but this was equally contentious. In his three-volume book, the definition of 2 comes at the end of volume I, and $1 + 1 = 2$ is proved on page 86 of volume II! There were more strange discoveries on numbers with Cantor's theory of transfinite numbers and different sizes of infinity. The chapter ends with Hilbert's ambitious project to put the whole of mathematics on a sound footing, but this was ruined by Gödel's shattering discoveries.

The final chapter of the book is on *Chaos Theory*, one of the new branches of Mathematics which challenges the deterministic, clockwork universe of Newton. Its beginnings were in 1886 when King Oscar II of Sweden offered a prize to solve the problem of stability of the solar system. Henri Poincaré, working on the three-body problem, realised the complexity of the problem; his work led to the development of chaos theory. In 1926, a Dutch engineer while simulating the heart using an electronic circuit realised that under certain conditions the resulting oscillations were irregular. However, it was another forty years before chaos theory began to be seriously studied. Meteorologist Edward Lorenz set out to model the atmospheric convection by approximating the complex equations with much simpler ones. He discovered that if the initial conditions varied even slightly, the differences became amplified and the final solutions looked very different from each other; this came to be known as the 'butterfly effect'. Parallel to these developments, a group of mathematicians towards the beginning of the 20th century were coming up with bizarre shapes: a curve that fills an entire space (Peano and Hilbert), a curve that crosses itself at every point (Sierpinski), a curve of infinite length that encloses a finite area (Koch). In the 1960s, Benoit Mandelbrot realised that these monstrous shapes reflected irregularities in nature and came up with the notion of a *fractal*. Mandelbrot was able to show many examples of fractals in nature, and in unexpected contexts like stock market prices.

Ian Stewart's book is an excellent resource for teachers who want to inspire their students; it can be equally enjoyed by students at the +2 level. It need not be read in a linear way. One can dip into any part of the book and make sense of it without having read earlier portions. There are many applications interspersed through the book. It is also a good read for those who wish to develop a coherent picture of modern mathematics as a whole, in terms of how the fundamental ideas relate to each other.

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The Closing Bracket . . .

The history of Mathematics has been irrevocably linked with the history of mankind. Mathematics has for years been the common language for classification, representation and analysis. Learning mathematics is an integral part of a child's education. Yet, for many reasons, it is traditionally perceived as a difficult subject and is a cause of distress for many students.

One of the reasons for this is, that the emphasis in mathematics learning is on rote memorization of procedures, paper-pencil-drills, manipulation of symbols and on learning tricks to solve problems. This is not to undermine the importance of procedures in mathematics learning but an overemphasis on procedures leads to a lack of relevance for the student who perceives the subject matter as abstract and far removed from everyday life. At the school level, few opportunities are provided in the classroom for visualization and exploration. In fact in the vast majority of classrooms in the country there remains a significant gap between content and pedagogy.

This brings us to the question of technology which has been the buzzword for most aspects of life, today. One of the most fundamental impacts of technology perhaps has been its contribution in expanding the boundaries of knowledge in almost every field. And in this age of innovations, is it any wonder that technology should reach the classroom, especially the mathematics classroom? After all if we agree that learning entails freedom – freedom to explore, experiment, question and visualize, then for all these modes of learning, technology, if properly used, can be a great enabler. It is therefore imperative that new and innovative methods of teaching enabled by technology increasingly find relevance in today's mathematics classroom.

World over, the advent of technological tools – handhelds in the form of graphic calculators, or computer software such as computer algebra systems (CAS), dynamic geometry software (DGS), spreadsheets and others have brought about a change in the way mathematics is taught. These tools have been evolving and over the years have succeeded in enabling the processes of visualization, exploration and experimentation. Many research studies in mathematics education have pointed to the positive impact of technology, highlighting its role in visualization of concepts, in exploration and discovery, in promoting multi-representational approach, in focusing on applications, in redefining the teacher's role and in helping sustain students' interest.

However, in India, the integration of technology especially in mathematics teaching has met with much resistance and scepticism. People have expressed the opinion that use of technology will increase the student's dependence on the tool and will be a detriment to the development of her thinking skills. She will no longer be able to do the paper-pencil calculations by hand. This argument is flawed since technology should not be used as a means to replace paper-pencil methods. Rather it should be used to develop the students' mathematical thinking by giving her