Approximate Constructions of Certain Angles

ClassRoom

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On the Facebook page of AtRiA, a reader (Surojit Shaw; see https://www.facebook.com/photo.php?fbid=787370044738665&set=p.787370044738665&type=3) posted the following comment in which he proposed constructions of certain angles (the words have been changed slightly, but the meaning is unaltered):

> *Using a compass, I construct a 60° angle EAB using 6 cm as the radius (see Figure 1; $AB = AE = BE = 6$ cm). Now for a 40° angle I take 4 cm as radius, put the compass point at B, draw an arc to cut the arc at C and join AC; $\measuredangle CAB$ will then be 40°. Similarly, for a 50° angle, I take 5 cm as the radius and repeat the same procedure ($BD = 5$ cm); $\measuredangle DAB$ will then be 50°. For a 8° angle, I take 0.8 cm, and so on. In this way we can construct other angles as well.*

The post invites us right away to try out the procedure using *GeoGebra*! We in turn invite the reader to do so and to explore the degree of accuracy of this construction.

The post is also instantly provocative; it seems to suggest that one can construct virtually any angle using a compass and a straightedge! Though the post mentions actual measurements (thus requiring a marked ruler), an unmarked straightedge would

Keywords: *approximate, angle, construction, compass*

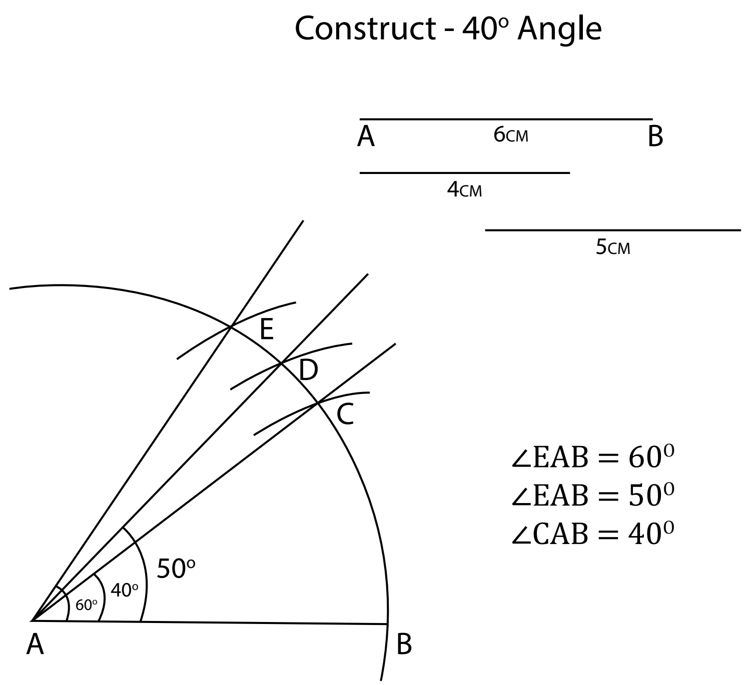


Figure 1

suffice. For: a 4 cm length is $2/3$ of a 6 cm length; and one can construct $2/3$ of a given line segment using only a compass and an unmarked straightedge.

In essence, the method may be described as follows. We draw a line segment AB with length a cm (in the FB post quoted above, we have $a = 6$) and then draw an arc centred at A, with the same radius a (Figure 2). We now wish to draw a ray AC such that $\measuredangle CAB$ has some desired measure t°. To do so, we measure off the length $t/10 \times a/6 = ta/60$ cm on the compass, lay the compass point at vertex B and draw an arc with this radius to cut the earlier arc at point C. (Why the fraction $t/10$? Examine the algorithm: for an angle of 50° he uses a radius of 5 cm, for an angle of 40° he uses a radius of 4 cm, and so on.) The claim then is that $\measuredangle CAB$ has the desired measure.

In Figure 2, the measure of $\measuredangle CAB$ is *supposed* to be t°. Let its actual measure be x°. To find the relationship between x and t, note that $\triangle CAB$ is isosceles, with $AB = AC$. Let M be the midpoint of segment CB; then AM is perpendicular to BC, so $\measuredangle MAB = x^\circ/2$, and $BM = ta/120$ cm. Hence we have:

$$\sin\frac{x^\circ}{2} = \frac{BM}{AB} = \frac{ta/120}{a} = \frac{t}{120},$$

$$\therefore \quad \frac{x}{2} = \frac{180}{\pi}\ \arcsin\frac{t}{120}, \tag{1}$$

and so:

$$x = \frac{360}{\pi}\ \arcsin\frac{t}{120}. \tag{2}$$

(Note: In equality (1), the multiplicative factor $180/\pi$ has been inserted because we are measuring the angle in degrees and not radians.)

We have found the desired relationship: $x = (360/\pi)\ \arcsin(t/120)$. This allows us to compute x for any given t. The table below has been computed using this formula.

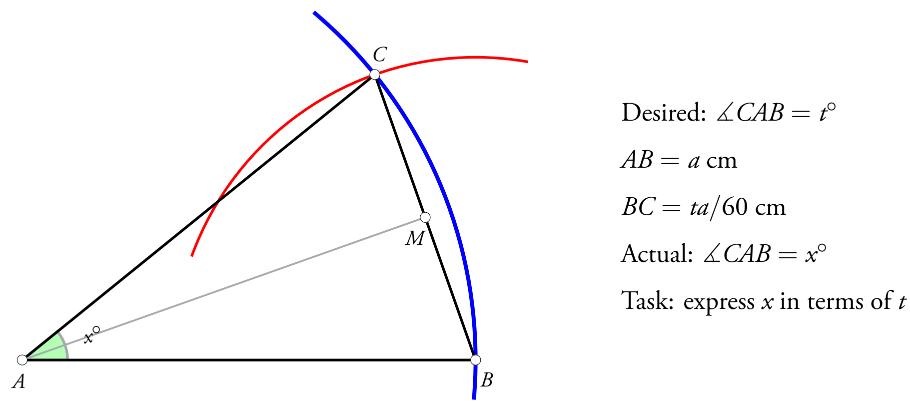


Figure 2

t	10	20	30	40	50	60	70	80	90
x	9.56	19.19	28.96	38.94	49.25	60	71.37	83.62	97.18

The results are striking. We observe that x is quite close to t for a good many values; and when $t = 60$, x and t are exactly equal to each other. For values of t beyond 60, however, the discrepancy between the two values grows steadily larger.

Figure 3 shows the graphs of both $x = (360/\pi)\ \arcsin(t/120)$ and $x = t$. Observe the closeness of the two graphs, especially for values of t between 0 and 60.

Mathematical essentials of the approximation

The approximation

$$t \approx \frac{360}{\pi} \arcsin \frac{t}{120} \qquad (0 \leq t \leq 60) \tag{3}$$

can be written in other ways that allow us to analyse it mathematically. We first write it as:

$$\sin \frac{\pi\, t}{360} \approx \frac{t}{120}, \qquad 0 \leq t \leq 60; \tag{4}$$

or, replacing t by $2t$ on both sides (the reason for doing this will become clear in a moment) and simplifying:

$$\sin \frac{\pi\, t}{180} \approx \frac{t}{60}, \qquad 0 \leq t \leq 30. \tag{5}$$

Now we have:

$$\frac{\pi\, t}{180} \text{ radians} = t^\circ.$$

Hence the proposed approximation is equivalent to the following assertion:

$$\sin t^\circ \approx \frac{t}{60}, \qquad 0 \leq t \leq 30; \tag{6}$$

or, switching back to radian measure:

$$\sin t \approx \frac{3t}{\pi}, \qquad 0 \leq t \leq \frac{\pi}{6}. \tag{7}$$

Note that the approximation is exact for $t = 0$ and for $t = \pi/6$ (i.e., for 0° and for 30°).

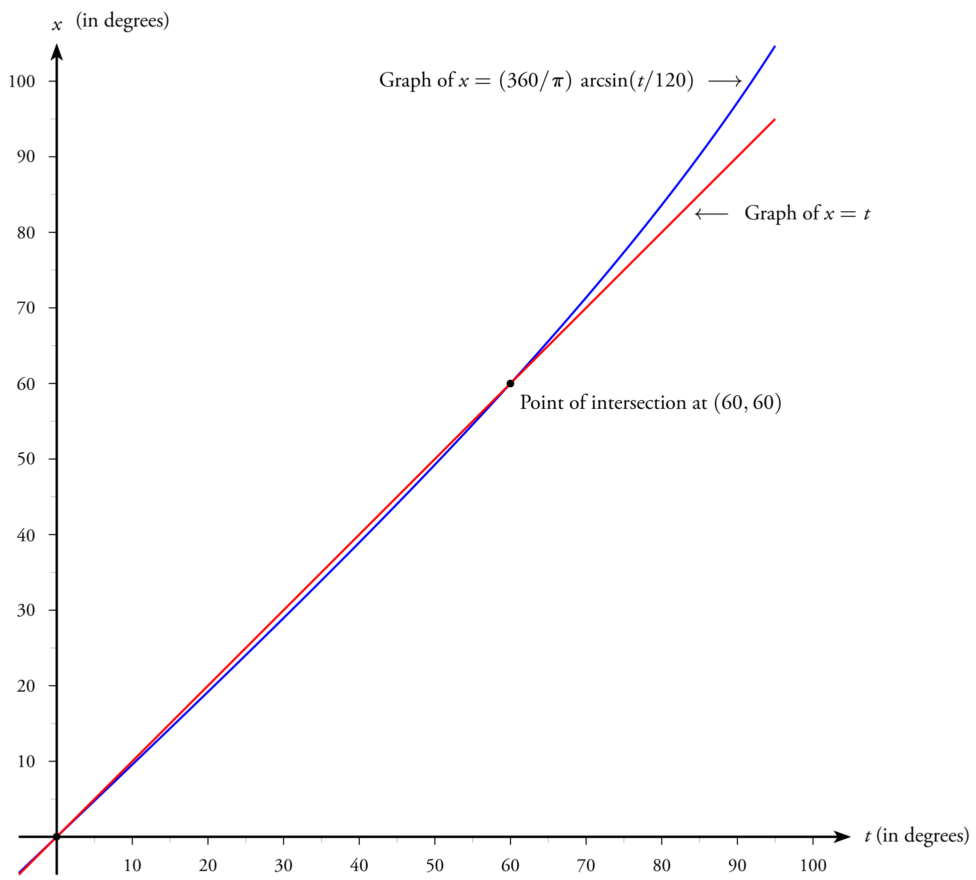


Figure 3

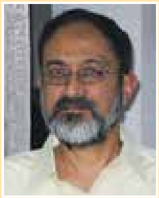
Error analysis. It is of interest to find out at which point in the interval I from 0 to $\pi/3$ the approximation is the worst. Let

$$f(t) = \sin t - \frac{3t}{\pi}, \quad 0 \le t \le \frac{\pi}{3}.$$

Then:

$$f'(t) = \cos t - \frac{3}{\pi}, \qquad f''(t) = -\sin t.$$

Using tables or a scientific calculator, we find that the acute angle whose cosine is $3/\pi$ is roughly 0.3014 radians, or roughly 17.27°; and, of course, $-\sin 17.27^\circ < 0$. For this value of t, therefore, $f(t)$ attains its maximum value within the interval I. The discrepancy between the two functions at this value of t is 0.00904. This represents a 3% error. Note that within I, $f(t)$ is consistently non-negative. So the function under study consistently overestimates the true value.



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