How to Prove It

In this episode of "How To Prove It", we prove a striking theorem first discovered by Ptolemy. We then discuss some nice applications of the theorem.

ClassRoom

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In this article we examine a famous and important result in geometry called *Ptolemy's Theorem*. Here is its statement (see Figure 1):

Theorem 1 (Ptolemy of Alexandria). *If ABCD is a cyclic quadrilateral, then we have the following equality:*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \tag{1}$$

In words: "The sum of the products of opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals."

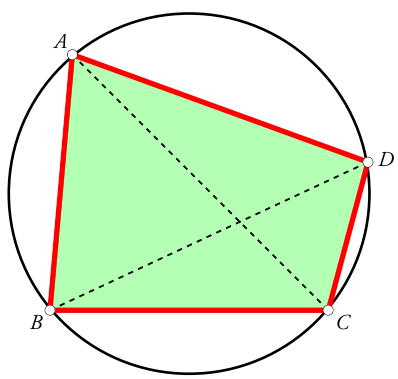


Figure 1. Cyclic quadrilateral and Ptolemy's theorem

Keywords: *Ptolemy, similar triangle, power of a point*

A 'pure geometry' proof. To prove the theorem presents a challenge. The difficulty lies in the fact that neither side of the equality $AB \cdot CD + AD \cdot BC = AC \cdot BD$ seems to *mean* anything. Terms like $AB \cdot CD$ and $AD \cdot BC$ suggest areas; but of what? There is nothing in the figure that yields a clue. So we try a different approach. We write the equality to be proved as

$$\frac{AB \cdot CD}{BD} + \frac{BC \cdot AD}{BD} = AC. \tag{2}$$

Have we made progress by writing it this way? Perhaps. Now the equality to be proved is a relation between *lengths*. Can we find or construct two segments whose lengths together yield the length of AC?

The expressions on the left ($AB \cdot CD/BD$ and $BC \cdot AD/BD$) suggest that we must look for or construct suitable pairs of similar triangles. Indeed, the form $AB \cdot CD/BD$ suggests that we should construct a triangle similar to $\triangle ABD$, and moreover that this (yet to be constructed) triangle should have CD for a side. Noting that $\angle ABD = \angle ACD$ we ask: what if we locate a point E on AC such that $\triangle ABD \sim \triangle ECD$? Then we would have $AB/BD = EC/CD$, giving $EC = AB \cdot CD/BD$; just what we want! Now we have a clue on how to proceed. Figure 2 shows the construction.

Locate a point E on diagonal AC such that $\angle CDE = \angle ADB$ (the two angles are marked with a bullet in Figure 2). Now consider $\triangle CDE$ and $\triangle ADB$. Since $\angle ECD = \angle ABD$ by the angle property of a circle, and $\angle CDE = \angle ADB$ by design, we have $\triangle CDE \sim \triangle ADB$. Hence:

$$\frac{EC}{CD} = \frac{AB}{BD}, \qquad \therefore\ EC = \frac{AB \cdot CD}{BD}. \tag{3}$$

With reference to the same figure (redrawn as Figure 3) we also have $\triangle DAE \sim \triangle DBC$, because $\angle DAE = \angle DBC$ and $\angle ADE = \angle BDC$.

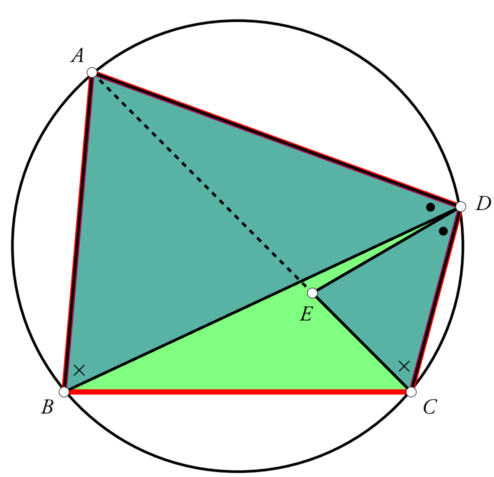
Hence:

$$\frac{BC}{BD} = \frac{AE}{AD}, \qquad \therefore\ AE = \frac{BC \cdot AD}{BD}. \tag{4}$$

Adding (3) and (4) we get, since $AE + EC = AC$:

$$AC = \frac{AB \cdot CD}{BD} + \frac{BC \cdot AD}{BD}, \qquad \therefore\ AC \cdot BD = AB \cdot CD + BC \cdot AD, \tag{5}$$

as was to be proved. □



Locate point E on diagonal AC such that $\angle CDE = \angle ADB$. Then $\triangle CDE \sim \triangle ADB$.

Figure 2. Construction of an appropriate point E on diagonal AC

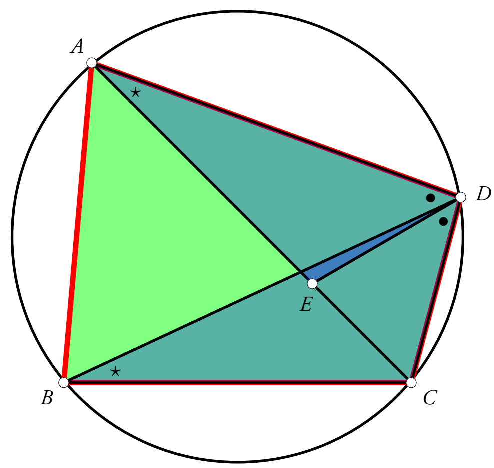


Figure 3. Another pair of similar triangles

You will agree that this is a very elegant proof (it is the proof given by Ptolemy), but it would not be easy to find it on one's own.

It turns out that Ptolemy's theorem can be proved in many different ways. Of particular interest are the following: (i) a proof using complex numbers, (ii) a proof using vectors, (iii) a proof based on a geometrical transformation called 'inversion'. We will have occasion to study these different ways in later articles.

A few elegant applications of Ptolemy's theorem

We showcase below three pleasing applications of the theorem proved above. The first one is an elegant result relating to an equilateral triangle.

Theorem 2. *Let ABC be an equilateral triangle, and let P be any point on the circumcircle of the triangle. Then the largest of the distances PA, PB, PC is equal to the sum of the other two distances.*

The theorem is illustrated in Figure 4. Note that P is located on the minor arc BC, i.e., it lies between the points B and C. The theorem now asserts that $PA = PB + PC$.

The proof is simplicity itself. Consider the cyclic quadrilateral $PBAC$. Apply Ptolemy's theorem to it; we get:

$$PA \cdot BC = PB \cdot AC + PC \cdot AB. \tag{6}$$

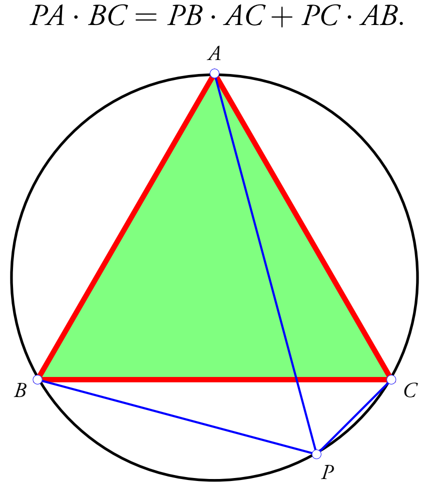


Figure 4. Application of Ptolemy's theorem to an equilateral triangle

But $AB = BC = CA$. The equal factor present in all three products in (6) can be cancelled, and we are left with the desired relation, $PA = PB + PC$. □

Remark. Theorem 2 can also be proved using the following trigonometric identity: for all angles θ (measured in degrees),

$$\sin(60 - \theta) + \sin\theta = \sin(60 + \theta). \tag{7}$$

The next result that we describe refers to a regular pentagon. If you examine such a pentagon, you will notice that it has five diagonals all of which have the same length.

Theorem 3. *Given a regular pentagon with side a, let its diagonals have length d. Then we have the following relation:*

$$a^2 + ad = d^2. \tag{8}$$

In Figure 5, $ABCDE$ is a regular pentagon; its sides have length a and its diagonals have length d. We apply Ptolemy's theorem to the inscribed quadrilateral $BCDE$; we get:

$$BC \cdot DE + BE \cdot CD = BD \cdot CE. \tag{9}$$

That is, $a^2 + ad = d^2$, as required. □

If we write $x = d/a$ (i.e., x is the ratio of the diagonal to the side of a regular pentagon), then the above equation yields: $x^2 = x + 1$. Solving this we get:

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

The negative sign clearly cannot hold, since x is positive; hence we have:

$$x = \frac{\sqrt{5} + 1}{2}. \tag{10}$$

So the ratio of the diagonal to the side of a regular pentagon is the Golden Ratio!

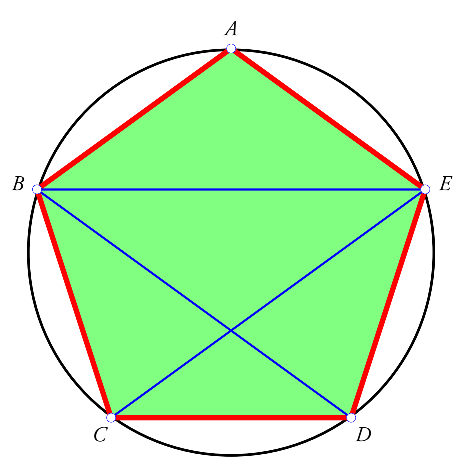


Figure 5. Application of Ptolemy's theorem to a regular pentagon

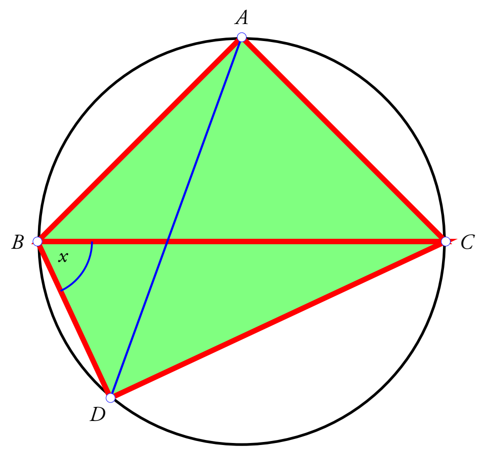


Figure 6. Application of Ptolemy's theorem to a trig inequality

Our third result is a trigonometric inequality which would seem difficult to prove using purely geometric methods. We shall show that for any acute angle x,

$$\sin x + \cos x \leq \sqrt{2}. \tag{11}$$

Figure 6 shows a circle with unit radius, a diameter BC, an isosceles right-angled triangle ABC with BC as hypotenuse, and a right-angled triangle DBC with BC as hypotenuse and with one acute angle equal to x; vertices A and D lie on opposite sides of BC.

We apply Ptolemy's theorem to the quadrilateral $ABDC$:

$$AB \cdot CD + AC \cdot BD = AD \cdot BC.$$

Since $AB = AC = \sqrt{2}$, $BD = 2\cos x$, $CD = 2\sin x$, $BC = 2$, we get:

$$2\sqrt{2}\,\sin x + 2\sqrt{2}\,\cos x = 2\,AD.$$

Now note that AD is a chord of the circle and so does not exceed in length the diameter of the circle; hence $AD \leq 2$. This yields $2\sqrt{2}\,\sin x + 2\sqrt{2}\,\cos x \leq 4$, and therefore:

$$\sin x + \cos x \leq \sqrt{2}, \tag{12}$$

as claimed. □

In the next episode of "How To Prove It" we shall showcase a few more applications of Ptolemy's theorem, and also prove an inequality version of the theorem.

References

1. A. Bogomolny, "Ptolemy's Theorem" from *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/proofs/ptolemy.shtml, Accessed 03 September 2016
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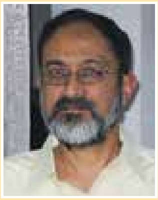

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