Fractal Constructions Leading to Algebraic Thinking

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This article describes how pre-service teachers explored fractal constructions using pictorial, numerical, symbolic and graphical representations while studying the topic geometric sequences in the Algebra unit of their mathematics course. By engaging with meaningful generalization tasks which required both explicit and recursive reasoning, they gained an insight into fractal geometry and also developed their algebraic thinking.

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Introduction

One of the foundational aspects of developing algebraic thinking is the ability to generalize. Research describes two kinds of generalization (Kinach, 2014), namely, *generalization by analogy* and *generalization by extension*. Generalization by analogy refers to observing a pattern, extending a sequence to the next few terms and being able to relate a particular term of the sequence to its previous terms. This kind of generalization requires *recursive thinking*. Generalization by extension, on the other hand, refers to writing a formula for the nth term of a sequence – which requires *explicit thinking*. Both kinds of generalization require abstraction and form the core of algebraic thinking. In fact generalization is a skill which is required at various stages of the school mathematics curriculum. However, guiding students through the process of generalization can be quite challenging and teachers must be familiar with tasks which can create a context for generalization.

Keywords: *Algebra, generalization, analogy, extension, recursive, explicit, Sierpinski triangle, self-similarity, fractal construction, infinite, geometric progression*

In this article we shall highlight that the topic of fractals provides an authentic context for engaging in generalization tasks giving ample opportunity for developing recursive and explicit thinking. According to Kinach (2014), *calculating the area and perimeter of the growing pattern of red triangles in the famous Sierpinski triangle and then expressing a formula for the area and perimeter for any iteration …are more advanced examples of generalization by analogy and extension.* (p.443)

We shall describe a module where 30 students of a pre-service teacher education programme explored fractals as a part of the Algebra unit in their *Core Mathematics* course. The primary goal was to engage them in exploring various patterns within fractal constructions through pictorial, tabular, symbolic and graphical representations, and to make connections between these representations. We will highlight that while going through the module they developed an insight into the nature of fractal geometry and engaged in meaningful generalization tasks emerging from the construction process. Mathematics curricula in many countries have emphasised the importance of developing algebraic thinking and the same has been articulated in the *Principles and Standards for School Mathematics* as 'expectations in the Algebra Standard' that students in grades 9 – 12 should be able to

- Generalize patterns using explicitly and recursively defined functions,…use symbolic algebra to represent and explain mathematical relationships;…[and] use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts. (NCTM 2000, p. 296)

The position paper *Teaching of mathematics* of the National Curriculum framework (NCF) 2005 (National Council for Educational Research and Training [NCERT], 2005) also articulates the importance of developing algebraic skills in the secondary school stage

- Algebra…is developed at some length at this stage. Facility with algebraic manipulation is essential, not only for applications of mathematics, but also internally in mathematics. Proofs in geometry and trigonometry show the usefulness of algebraic machinery. It is important to ensure that students learn to geometrically visualize what they accomplish algebraically. (NCF 2005, p. 17)

The fractal investigations – a background

The module was conducted by the author with 30 first year students of a pre-service teacher education programme, as a part of the algebra unit of their *Core Mathematics* course. The focus of this course is to enable the student teacher to enhance her content knowledge of the school mathematics curriculum. 12 out of the 30 students who went through this module had studied mathematics in school up to grade 10 and the rest had studied mathematics up to grade 12. Prior to the module, students had recapitulated their knowledge of arithmetic sequences, exponents and had been introduced to the concept of geometric sequences. The author (who was also their teacher) decided to use fractal constructions to enhance their understanding of geometric sequences.

Understanding fractal constructions through multiple representations

Being able to work with a variety of representations such as tables, pictures, graphs and abstracting their interrelationships are an essential aspect of developing algebraic reasoning. In this section we shall describe how students used pictorial, tabular, symbolic and graphical representations to explore fractal constructions.

Pictorial representations led to understanding of self-similarity

In the very first session of the module, students were introduced to the Sierpinski triangle construction. The construction process was briefly explained by the teacher. An equilateral triangle (stage 0) was drawn and cut out from a sheet of paper. The mid-points of the sides were joined, to obtain four smaller triangles and the centre triangle was removed. This piece with a triangular 'hole' was referred to as stage 1. Students observed that stage 1 comprised three identical smaller copies of stage 0 (each copy was a smaller

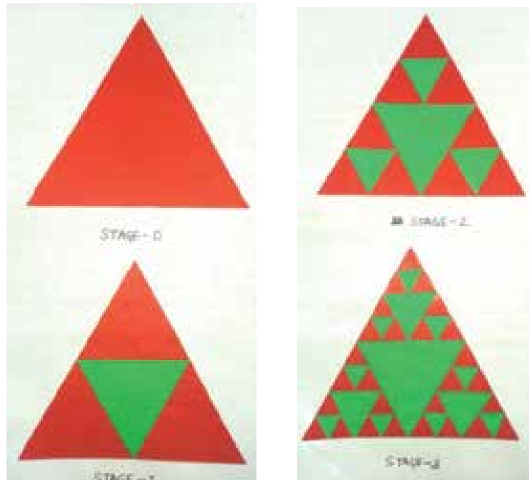


Figure 1. Stages 0 to 3 of the Sierpinski triangle as depicted by a student.

equilateral triangle). The process of creating smaller equilateral triangles and removing the centre triangle was repeated on the three smaller triangles of stage 1 to obtain stage 2. Figure 1 shows stages 0,1,2 and 3 as obtained by a student who preferred to use a combination of red and green triangles. The green triangular portions represent the triangular 'holes'.

After the construction process was over, some time was spent on discussing students' observations. A few students said that the process could 'go on forever' although many could not describe what higher stages would look like. One student commented that 'the number of triangular holes will go on increasing' referring to the parts which are being removed. A majority of students agreed that the number of triangles 'will increase at every stage and each triangle will also get smaller in size'.

To give a direction to their observations, students were assigned two tasks. The first task required them to count the number of shaded triangles in stages 0 to 3 and predict the number for stages 4 and 5. They were required to find a rule for the number of shaded triangles at the nth stage.

In the second task, they had to find a rule for shaded area at the various stages and also at the nth stage (given that the area of the equilateral triangle at stage 0 is 1 square unit). At this point the teacher helped students to make the observation that stage 1 has three smaller copies of stage 0. Similarly stage 2 has three smaller copies of stage 1 and nine still smaller copies of stage 0. This idea of identifying smaller copies of previous stages in subsequent stages was introduced as self-similarity. Figure 2 was used by the teacher to explain this idea.

Numerical representations led to generalization by extension

Task 1 was easily done by all students as they observed that the number of shaded triangles at each stage was 'a power of 3' and using a

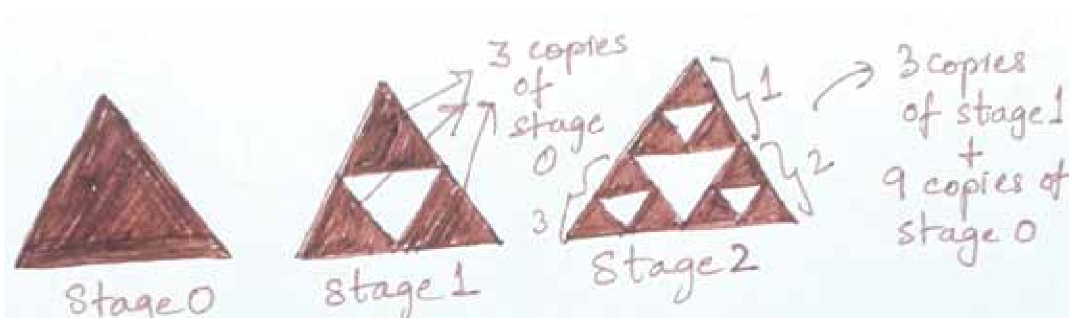


Figure 2. The idea of self-similarity -- finding scaled down copies of previous stages in a given stage.

multiplying factor of 3, they came up with the geometric sequence $1, 3, 3^2, 3^3, \ldots\ldots.$ However, the second task posed a challenge for a few students. While they concluded that the shaded area at stage 1 is $3/4$ units (since only three of the 4 smaller equilateral triangles were shaded), they were unable to extend the idea to stage 2. A few students pointed out 'the shaded area at stage 1 is being divided into 4 equal parts in stage 2 and one of these parts is being removed' thus leading them to conclude that the shaded area at stage 2 is $3/4$ of $3/4$, that is, 9/16 or $(3/4)^2$. This idea was taken up by others and extended to the fact that the multiplying factor in the sequence of shaded areas was 3/4. Finally a majority of the class obtained the geometric sequence $1, \frac{3}{4}, \left(\frac{3}{4}\right)^2, \left(\frac{3}{4}\right)^3, \ldots\ldots\ldots$ to represent the shaded area at various stages. Students worked in pairs, reasoned about the number of shaded triangles and shaded area using their pictorial and tabular representations and arrived at the geometric sequences. These may be considered as examples of generalization by analogy. However, finding the formula for the nth stage entails generalization by extension. This required them to observe that the exponents of 3 and $3/4$ in the two sequences coincide with the stage number. With facilitation, students were able to conclude that the nth terms of the sequences were 3^n and $\left(\frac{3}{4}\right)^n$ respectively. This exercise led to two geometric sequences, one with common ratio 3 (greater than 1) and the other with common ratio $3/4$ (less than 1).

Symbolic representations

At this stage, the teacher tried to help students to make connections between their recursive and explicit reasoning. She introduced the following symbols and asked them to write the nth terms of the two sequences using these

S_n= number of shaded triangles at stage n,
A_n= Shaded area at stage n

Students had to relate the formula of stage n with that of stage n-1 for both sequences. The aim was to help them see the recursive relationships within each attribute (number of shaded triangles and shaded area) and to think of the nth terms of the sequences as independent expressions which they could manipulate.

For the number of shaded triangles at every stage, students obtained the generalized formula $S_n = 3^n$. Writing the recursive relation $S_n = 3 \times S_{n-1}$ however, took some scaffolding. The teacher had to emphasise that $S_1 = 3 \times S_0$ and $S_2 = 3 \times S_1$ to help them see the relation.

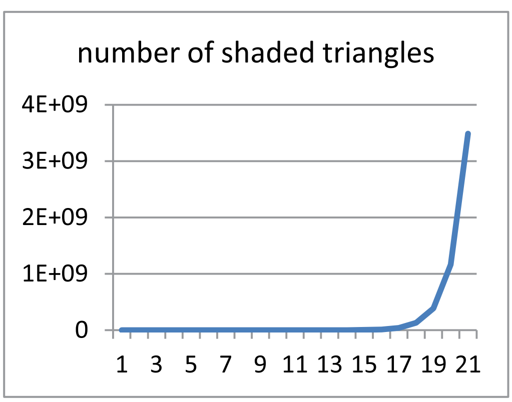
For shaded area, students came up with the recursive and explicit formulae, $A_n = \frac{3}{4} \times A_{n-1}$ and $A_n = \left(\frac{3}{4}\right)^n$ more easily. At this point, the teacher asked them to express the self-similarity of the Sierpinski triangle using the same ideas. After some facilitation, many students could articulate the idea that stage n has three copies of stage n-1, 9 copies of stage n-2, etc. For the teacher, this was a high point of the class, as it convinced her that students had succeeded in generalizing the Sierpinski triangle construction through multiple representations.

A spreadsheet exploration of the fractal constructions

Finally students were assigned the task of describing what would happen as n, the number of stages, approached infinity. They conjectured that the number of shaded triangles would 'become very large' and some used the phrase 'will approach infinity'. For the shaded area, many students said it would get 'smaller and smaller'. To help them visualise this numerically, the students were encouraged to explore these sequences on MS Excel by generating values up to stage 20 (see figure 3). For example, the first column shows 'n' the stage number (up to 20); the second column shows the number of shaded triangles, obtained by entering 1 in the first cell (say C3) and $\boxed{= \text{C3*3}}$ in cell C4. The sequence of shaded areas was similarly obtained in the third column. Graphing the sequences revealed that the number of shaded triangles was growing very rapidly whereas the shaded area was approaching 0. Thus Excel played a pivotal role in helping students visualize the fractal construction process, numerically and graphically. It is impossible to draw the Sierpinski triangle after stage 4 or 5. However, using Excel students could visualize the growth process at higher stages.

stage	no of shaded triangles	shaded area
0	1	1
1	3	0.75
2	9	0.5625
3	27	0.421875
4	81	0.31640625
5	243	0.237304688
6	729	0.177978516
7	2187	0.133483887
8	6561	0.100112915
9	19683	0.075084686
10	59049	0.056313515
11	177147	0.042235136
12	531441	0.031676352
13	1594323	0.023757264
14	4782969	0.017817948
15	14348907	0.013363461
16	43046721	0.010022596
17	129140163	0.007516947
18	387420489	0.00563771
19	1162261467	0.004228283
20	3486784401	0.003171212

(i)



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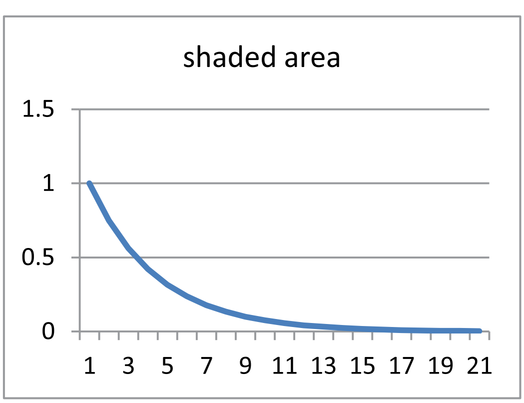

(iii)

Figure 3. Numerical and graphical representations of the geometrical sequences arising from the Sierpinski triangle construction in MS Excel.

By the end of the first two-hour session, students had been introduced to the nature of fractal constructions, meaning of self – similarity and had quantified the patterns emerging from the construction process. They had succeeded in exploring the Sierpinski triangle using multiple representations, made connections between these representations and had engaged in explicit as well as recursive reasoning. The exploration in MS Excel led to a 'big picture' understanding of the Sierpinski triangle.

In the second session, students asked if they could extend the idea of the Sierpinski triangle construction to a square. Their efforts, interestingly, led to the Sierpinski square carpet (see figure 4). Here the construction process requires each side of the square piece (stage 0) to be trisected. When points of trisection of opposite sides are joined, 9 smaller squares are created. To obtain stage 1, the centre square is removed and 8 shaded squares are obtained. The same process is repeated on the 8 smaller shaded squares to obtain stage 2. The construction process was done quite easily by most students. They used a dotted grid paper, so as to make the trisection process easy. Without even being asked, students tried to predict the geometric sequences which would emerge by counting the number of shaded squares and shaded area at each stage. It was not difficult for them to conclude that the number of shaded squares led to the geometric sequence $1, 8, 8^2, 8^3, \ldots\ldots.$ which could be represented explicitly and recursively using the relations $S_n = 8^n$ and $S_n = 8 \times S_{n-1}$. For the shaded area, a discussion among students led to the conclusion that the multiplying factor was 8/9. The geometric sequence $1, \frac{8}{9}, \left(\frac{8}{9}\right)^2, \left(\frac{8}{9}\right)^3, \ldots\ldots\ldots$ with explicit and recursive relations $A_n = \left(\frac{8}{9}\right)^n$ and $A_n = \frac{8}{9} \times A_{n-1}$ was obtained. Students identified self-similarity within the Sierpinski carpet by extending the idea from the Sierpinski triangle.

Students' explorations took an interesting turn at this stage in the module. Those who had studied geometric sequences in grade 11 (*Sequences and Series* is a topic in the grade 11 syllabus as prescribed by the Central Board of Secondary

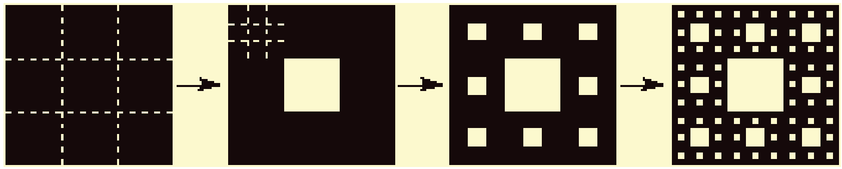

Figure 4. Stages 0 to 3 of the Sierpinski square carpet construction (retrieved from http://www2.edc.org/makingmath/mathprojects/pascal/pascal_warmup.asp)

Education (CBSE)) during their school days, wanted to know if the meaning of the formula for S_∞ could be visualised in the context of the geometric sequences arising out of the Sierpinski constructions. What would happen if, for example, they took the sum (to infinity) of the geometric progression $1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots \infty$? The teacher found this to be an exciting opportunity. She encouraged students to explore the cumulative sum of shaded areas represented by the above progression in MS Excel. Figure 5 shows the Excel output where the first column represents the stages, the second column, terms of the sequence of shaded areas, and third column, the cumulative sum of areas. Indeed by the 20^{th} term the cumulative sum approaches a fixed value, 4. This was verified by students using the formula $S_\infty = \frac{a}{1-r}$, where a is the initial term and r, the common ratio of the geometric progression. Of course, this is applicable only when $|r| < 1$. Thus $S_\infty = \frac{1}{1-3/4} = \frac{1}{1/4} = 4$. Graphing the second and third columns (see figures 3 (iii) and 5 (ii) respectively) helped them to visualize the process. While the sequence of shaded areas was approaching 0 as n approached infinity, the cumulative sum of areas was approaching 4. The graphical representations led to an interesting discussion in the class. 'But how can we explain the infinite process leading to a fixed value?' some students asked. Another group of students, after some discussion, explained - 'the amount of area getting added at each successive stage is decreasing, so effectively, the total area is approaching a fixed value.' One student commented 'I had used the formula for S_∞ in school, but I never knew what it meant. Today it makes sense!' This was indeed the high point of the class! It was very satisfying to see that students, who had studied geometric sequences in school, now actually understood them and those who had not studied this topic earlier, had also learnt it in a meaningful way. In the beginning of the module students had relied more on pictorial and tabular representations to understand the fractal

stage	shaded area	sum of area upto stage n
0	1	1
1	0.75	1.75
2	0.5625	2.3125
3	0.421875	2.734375
4	0.31640625	3.05078125
5	0.237304688	3.288085938
6	0.177978516	3.466064453
7	0.133483887	3.59954834
8	0.100112915	3.699661255
9	0.075084686	3.774745941
10	0.056313515	3.831059456
11	0.042235136	3.873294592
12	0.031676352	3.904970944
13	0.023757264	3.928728208
14	0.017817948	3.946546156
15	0.013363461	3.959909617
16	0.010022596	3.969932213
17	0.007516947	3.97744916
18	0.00563771	3.98308687
19	0.004228283	3.987315152
20	0.003171212	3.990486364

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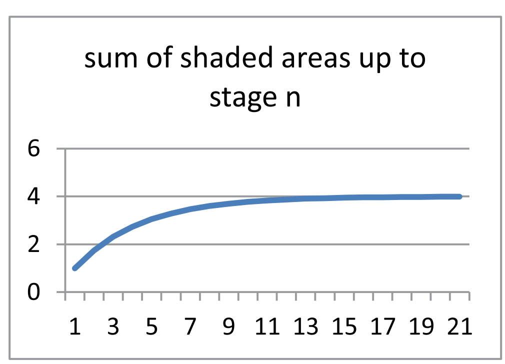
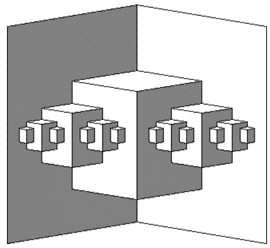


(ii)

Figure 5. Numerical and graphical representations of the cumulative sum of shaded areas of the various stages of the Sierpinski triangle in MS Excel.



(i)

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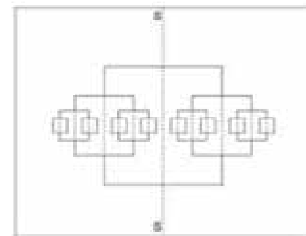
Figure 6. A fractal card obtained by repeated cutting and folding.



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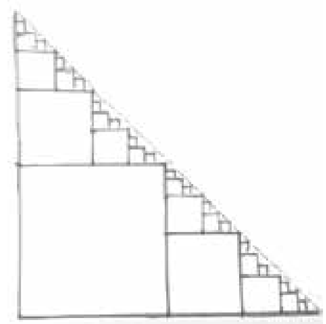

(ii)

Figure 7. The fractal card when flattened (i) and its cross-section (ii).

constructions, but later they made the transition to representing the same ideas using symbols, thus obtaining recursive and explicit formulae.

In the remaining part of the module, students learnt to make fractal cards and explored number patterns within them. For example, Figure 6 (i) shows a drawing of a fractal card obtained by repeated cutting and folding a rectangular sheet of paper and pushing out the 'cells'. Figure 6 (ii) shows a photograph of an actual card made by a student.

While exploring the card they identified several geometric sequences within the card. The number of 'pop up' cells at the various stages led to the sequence, $1, 2, 2^2, 2^3$,....... When the card is flattened (figure 7 (i)), the lengths of the cuts (horizontal lines) leads to the sequence $\frac{l}{2}, \frac{l}{4}, \frac{l}{8}, \frac{l}{16}, \ldots$ with the nth term $\frac{l}{2^n}$, where 'l' denotes the length of the rectangular sheet. The cross section of the card (figure 7 (ii)) reveals squares of reducing size. If the process is continued to infinity and the areas of the squares are added, we get the geometric progression $\frac{l^2}{16} + \frac{l^2}{32} + \frac{l^2}{64} + \cdots$. whose sum to infinity is $\frac{l^2}{8}$. In fact, as the process of cutting and folding is continued, smaller squares appear with each successive stage and these approach the hypotenuse of the right angled triangle whose area is $\frac{l^2}{8}$. Once again students encountered a process, where the sum of infinitely many terms of a geometric sequence actually approaches a fixed value. The output of the fractal card activity led to much excitement as the cards were very attractive and students tried to think of other kinds of cuts and folds which could lead to cards with fractal structure.

Conclusion

The module described in this article provided the pre-service teachers with the opportunity to visualize and explore geometric sequences through fractal constructions. Using multiple representations – pictures, tables, symbols and graphs, they generalised various attributes of fractals such as the Sierpinski triangle, Sierpinski carpet and fractal cards. The fractal constructions provided an authentic context to engage in recursive as well as explicit reasoning thus leading to meaningful generalisation of the fractals at higher stages. Further, Excel helped them to visualize the generalization process numerically, by generating values of the sequences at higher stages which could not be calculated manually and also by producing graphs which illustrated the behaviour of the attributes in the long run. It helped them to see that as 'n' increased, the sum of terms of particular geometric sequences approach fixed values, a concept which they were unable to visualize earlier. To summarise, the fractal explorations in the module helped the pre-service teachers to gain an insight into the nature of fractal geometry and familiarised them with a range of activities which can be easily integrated into classroom teaching at the secondary level. The module also highlighted the power of generalization in leading to algebraic reasoning.

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