Problem Corner

Problems for the Senior School

Problem Editors: Prithwijit De & Shailesh Shirali

PROBLEMS FOR SOLUTION

Problem V-3-S.1
Let $ABCD$ be a convex quadrilateral. Let P, Q, R and S be the midpoints of AB, BC, CD and DA respectively. What kind of a quadrilateral is $PQRS$? If $PQRS$ is a square, prove that the diagonals of $ABCD$ are perpendicular to each other.

Problem V-3-S.2
Let $ABCD$ be a convex quadrilateral. Let P, Q, R and S be the midpoints of AB, BC, CD and DA respectively. Let U and V be the midpoints of AC and BD, respectively. Prove that the lines PR, QS and UV are concurrent.

Problem V-3-S.3
There are 12 lamps, initially all OFF, each of which comes with a switch. When a lamp's switch is pressed, it's state is reversed, i.e., if it is ON, it will go OFF, and vice-versa. One is allowed to press exactly 5 different switches in each round. What is the minimum number of rounds needed so that all the lamps will be turned ON? (Hong Kong Preliminary Selection Contest 2015)

Problem V-3-S.4
The greatest altitude in a scalene triangle has length 5 units, and the length of another altitude is 2 units. Determine the length of the third altitude, given that it is integer valued.

Problem V-3-S.5
If the three-digit number $\overline{ABC}$ is divisible by 27, prove that the three-digit numbers $\overline{BCA}$ and $\overline{CAB}$ are also divisible by 27.

SOLUTIONS OF PROBLEMS IN ISSUE-V-2 (JULY 2016)

Call a convex quadrilateral *tangential* if a circle can be drawn tangent to all four sides. All the problems studied below have to do with this notion.

Solution to problem V-2-S.1
Let ABCD be a tangential quadrilateral. Prove: $AD + BC = AB + CD$.

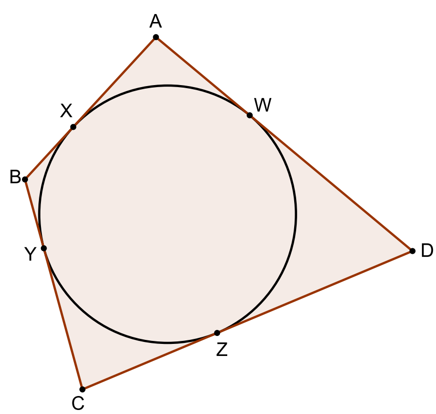

Figure 1

Let the circle touch the sides AB, BC, CD, DA at X, Y, Z, W respectively (Figure 1). Then $AX = AW$, $BX = BY$, $CY = CZ$ and $DZ = DW$. Therefore

$$\begin{aligned} AD + BC &= (AW + WD) + (BY + YC) = (AX + DZ) + (XB + ZC) \\ &= (AX + XB) + (DZ + ZC) = AB + CD. \end{aligned}$$

Solution to problem V-2-S.2
Let ABCD be a convex quadrilateral with $AD + BC = AB + CD$. *Prove that ABCD is tangential.*

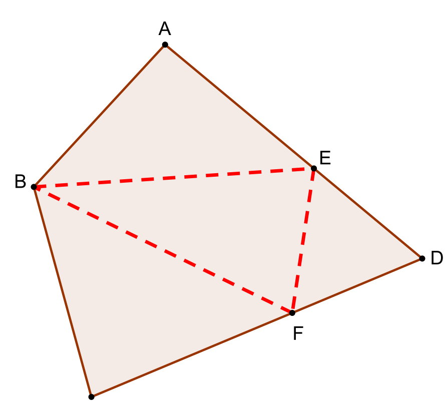

Figure 2

Suppose that $AD > AB$. Then $CD > BC$. Let E be a point on the segment AD such that $AE = AB$ and let F be a point on the segment CD such that $BC = CF$. Then it follows that $DE = DF$ and the triangles ABE, BCF and DEF are isosceles. The perpendicular bisectors of the sides BE, BF and EF of triangle BEF are concurrent at the circumcentre of BEF. But the perpendicular bisectors of BE, BF and EF are also the angle bisectors of $\measuredangle BAE$, $\measuredangle BCF$ and $\measuredangle EDF$. Thus three angle bisectors of the quadrilateral $ABCD$ are concurrent. This point of concurrency is equidistant from all four sides of the quadrilateral and hence is the centre of its inscribed circle. Therefore $ABCD$ is tangential. The case when $AD = AB$ is left as an exercise to the reader.

Solution to problem V-2-S.3

Place four coins of different sizes on a flat table so that each coin is tangent to two other coins. Prove that the quadrilateral formed by joining the centres of the coins is tangential. Prove also that the convex quadrilateral whose vertices are the four points of contact is cyclic. Is the circle passing through the four points of contact tangent to the sides of the tangential quadrilateral formed by joining the four centres of the coins?

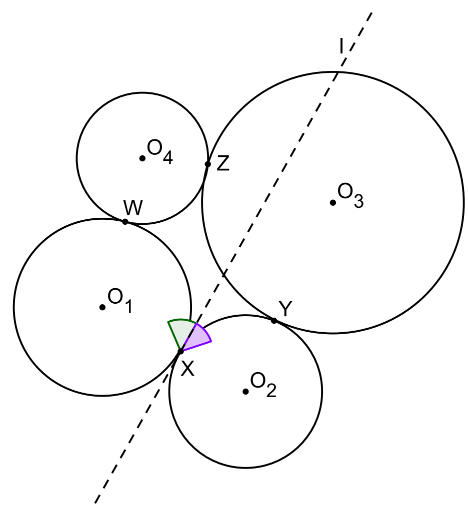

Figure 3

Let the radii of the coins be r_1, r_2, r_3, r_4 respectively. One pair of opposite sides of the quadrilateral formed by joining the centres of the coins are $r_1 + r_2$ and $r_3 + r_4$ whereas the other pair of opposite sides are $r_1 + r_4$ and $r_2 + r_3$. Evidently the sums of the lengths of the pairs of opposite sides are same and equal to $r_1 + r_2 + r_3 + r_4$. Thus the quadrilateral is tangential.

Name the coins C_1, C_2, C_3, C_4. Let the centres of the coins be O_1, O_2, O_3 and O_4 and let the point of contact of C_1 and C_2 be X, that of C_2 and C_3 be Y. Let Z and W be the points of contact of C_3 and C_4, and C_4 and C_1 respectively. Let l be the common tangent of C_1 and C_2 passing through X. Then the angle between WX and l is $\frac{1}{2}\measuredangle WO_1X$ and the angle between YX and l is $\frac{1}{2}\measuredangle XO_2Y$. Therefore

$$\measuredangle WXY = \frac{\measuredangle WO_1X + \measuredangle XO_2Y}{2}.$$

Similarly

$$\measuredangle WZY = \frac{\measuredangle WO_4Z + \measuredangle YO_3Z}{2}.$$

Therefore

$$\measuredangle WXY + \measuredangle WZY = \frac{\measuredangle WO_1X + \measuredangle XO_2Y + \measuredangle WO_4Z + \measuredangle YO_3Z}{2} = 180^\circ.$$

Hence the quadrilateral $XYZW$ is cyclic.

As regards the question of whether the circle through X, Y, Z, W is the incircle of quadrilateral $O_1O_2O_3O_4$, the answer is: *it need not be*. For "proof" please see Figure 4. (Actually, a diagram can never serve as a proof! So there is rather more to this question than meets the eye. We will offer a fuller analysis of the matter in the next issue, March 2017.)

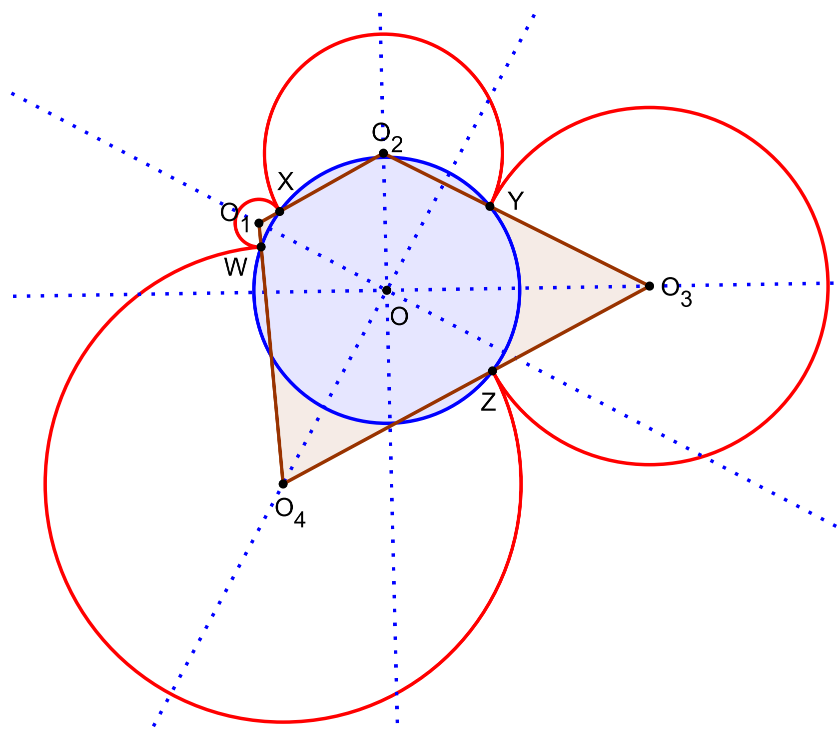

Figure 4

Solution to problem V-2-S.4

Let ABCD be a cyclic quadrilateral and let X be the intersection of diagonals AC and BD. Let P_1, P_2, P_3 and P_4 be the feet of the perpendiculars from X to BC, CD, DA and AB respectively. Prove that quadrilateral $P_1P_2P_3P_4$ is tangential.

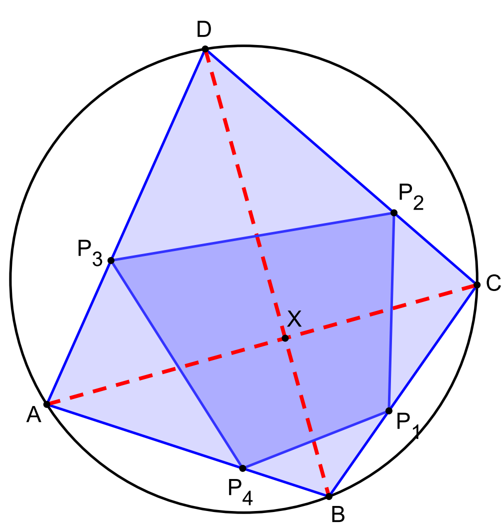

Figure 5

See Figure 5. Observe that the quadrilaterals P_4BP_1X and P_1CP_2X are cyclic. In quadrilateral P_4BP_1X, we have $\measuredangle P_4BX = \measuredangle P_4P_1X$. In quadrilateral P_1CP_2X, we have $\measuredangle P_2CX = \measuredangle P_2P_1X$. As $ABCD$ is cyclic,

$$\measuredangle P_4BX = \measuredangle ABD = \measuredangle ACD = \measuredangle P_2CX.$$

Therefore $\measuredangle P_4P_1X = \measuredangle P_2P_1X$. Thus XP_1 bisects $\measuredangle P_4P_1P_2$. Similarly it can be shown that XP_2, XP_3 and XP_4 bisect $\measuredangle P_1P_2P_3$, $\measuredangle P_2P_3P_4$ and $\measuredangle P_3P_4P_1$ respectively. Thus X is the point of concurrency of the angle bisectors $P_1P_2P_3P_4$ and is therefore equidistant from all four sides of the quadrilateral. Therefore X is the centre of the inscribed circle and $P_1P_2P_3P_4$ is tangential.

SOLUTIONS

NUMBER CROSSWORD

Page 23

1 2	2 3	4	0		3 1	0	4 8	9
	1			5 2			4	
	6 2	9		9		7 2	5	
8 6			9 4	4	4			10 1
11 5	2	8				12 5	9	4
0			13 5	14 1	2			4
	15 6	5		0		16 1	17 2	
	2			1			1	
18 3	1	7	7		19 9	8	0	1